A General Solution of a Closed-Form Space Resection

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ABSTRACT: A general solution of closed form space resection is proposed and described. When three object control points exist, a maximum of four solutions for the six exterior orientation parameters can be found without iteration and initial information. If a fourth point not lying on the critical surface exists, a unique solution can be achieved.

INTRODUCTION

As DEFINED BY Moffitt and Mikhail (1980), space resection is the process by which the spatial position and orientation of a photograph is determined based on the photogrammetric measurements of the images of ground control points appearing in the photographs. Conventionally, the spatial position and orientation of a photograph are represented by the three coordinates of the exposure station and three rotation parameters. They are called exterior orientation parameters. A closed solution means that there is no need for any initial values and iterations. A lot of effort has been made by photogrammetrists and mathematicians to obtain a closed solution. This topic is of interest to the photogrammetric community, especially for close-range industrial photogrammetry. In the latter case, appropriate approximate values of exterior orientation, which are hard to measure, are absolutely needed to make the bundle adjustment converge. The approach described in this paper can be used to obtain such approximate values. In addition, it is of great interest in robot vision application. A direct linear transformation (DLT) solution was proposed for this purpose by Abdel-Aziz and Karara (1971), but six object points are needed. The topic was discussed in Faugeras and Toscani, (1986), Ganapathy (1984), and Tsai (1987) for a closed-form solution or a two-step method. But, as was noted by Shih and Faig (1988), the general solution of a closed-form space resection with three (coordinates of the exposure station) or six (three rotation parameters as well) parameters is still lacking. Although there are some closed-form solutions, some require more object information, such as the DLT method while others assume that there exist some additional constraints, such as the assumption that the object plane is nearly parallel to the image plane (Rampel, 1979).

This paper describes a general solution of closed form space resection. A similar approach was suggested by Fischler and Bolles (1981). It dealt with the so-called location determination problem (LDP) for image analysis and included a general solution for obtaining the exposure station position and discussed the multiple-valued nature of the problem. Our own approach includes three main parts: First, a general solution to the three coordinates of the exposure station; second, a general solution to obtain the three rotation parameters (this approach is original and is based on the Pope-Hinsken algorithm [Hinsken, 1988]); and third, a discussion of the critical curve in space resection. It is well known that there is a critical circle in planar resection and a critical (cylindrical) surface in relative orientation, but the existence of a critical curve in space resection has apparently not been previously discussed. The third part of our paper will also comment in detail on some conclusions in Fischler and Bolles (1981).

A METHOD OF FINDING THE EXPOSURE STATION

In Figure 1, the image pyramid is shown. S is the exposure station; 1, 2, 3 denote object control points; 1′, 2′, 3′ denote the corresponding image points (not collinear). L_i and R_i denote corresponding spatial distances between pyramid points.

It should be noted that the sequence of points 1′, 2′, 3′ is counterclockwise. The image coordinate system with origin at S is used in the following computation:

Let x_i, y_i, $-f$ be the image coordinates, where f is the principal distance and is directed opposite to the z-coordinate direction; and X_i, Y_i, Z_i be the coordinates of object points in the image system.

Then

$$\begin{aligned} x_1 &= -f \times X_1/Z_1, \quad y_1 = -f \times Y_1/Z_1 \\ x_2 &= -f \times X_2/Z_2, \quad y_2 = -f \times Y_2/Z_2 \\ x_3 &= -f \times X_3/Z_3, \quad y_3 = -f \times Y_3/Z_3 \end{aligned} \tag{1}$$

$$(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 = L_3^2 \tag{2}$$

$$(X_2 - X_3)^2 + (Y_2 - Y_3)^2 + (Z_2 - Z_3)^2 = L_1^2 \tag{3}$$

$$(X_3 - X_1)^2 + (Y_3 - Y_1)^2 + (Z_3 - Z_1)^2 = L_2^2 \tag{4}$$

where L_1, L_2, L_3 are calculated by using the given coordinates of control points in the ground coordinate system.

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After introducing Equation 1 into Equations 2 to 4, it follows that

$$A \times Z_1^2 - D \times Z_1 \times Z_2 + B \times Z_2^2 = L_3^2 \tag{5}$$

$$B \times Z_2^2 - E \times Z_2 \times Z_3 + C \times Z_3^2 = L_1^2 \tag{6}$$

$$C \times Z_3^2 - F \times Z_3 \times Z_1 + A \times Z_1^2 = L_2^2 \tag{7}$$

where

$$\begin{aligned} A &= (x_1^2 + y_1^2 + f^2)/f^2 \\ B &= (x_2^2 + y_2^2 + f^2)/f^2 \\ C &= (x_3^2 + y_3^2 + f^2)/f^2 \\ D &= (2 \times x_1 \times x_2 + 2 \times y_1 \times y_2 + 2 \times f^2)/f^2 \\ E &= (2 \times x_2 \times x_3 + 2 \times y_2 \times y_3 + 2 \times f^2)/f^2 \\ F &= (2 \times x_3 \times x_1 + 2 \times y_3 \times y_1 + 2 \times f^2)/f^2 \end{aligned} \tag{8}$$

According to Equation 8, it is clear from the geometry in Figure 2 that we can get following relations:

$$\begin{aligned} A &= 1/\cos^2\Theta_{f1},\ B = 1/\cos^2\Theta_{f2},\ C = 1/\cos^2\Theta_{f3} \\ D &= 2 \times \cos\Theta_{12}/(\cos\Theta_{f1} \times \cos\Theta_{f2}) \\ E &= 2 \times \cos\Theta_{23}/(\cos\Theta_{f2} \times \cos\Theta_{f3}) \\ F &= 2 \times \cos\Theta_{31}/(\cos\Theta_{f3} \times \cos\Theta_{f1}) \end{aligned} \tag{9}$$

Assume that

$$\begin{aligned} D_1 &= D/\sqrt{A \times B} = 2 \times \cos\Theta_{12} \\ E_1 &= E/\sqrt{B \times C} = 2 \times \cos\Theta_{23} \\ F_1 &= F/\sqrt{C \times A} = 2 \times \cos\Theta_{31} \end{aligned} \tag{10}$$

From Figure 1, we have

$$\begin{aligned} Z_1 &= -R_1 \times \cos\Theta_{f1} = -R_1/\sqrt{A} \\ Z_2 &= -R_2 \times \cos\Theta_{f2} = -R_2/\sqrt{B} \\ Z_3 &= -R_3 \times \cos\Theta_{f3} \times -R_3/\sqrt{C} \end{aligned} \tag{11}$$

Substituting Equations 10 and 11 into Equations 5 to 7, we obtain the following equations:

$$R_1^2 - D_1 \times R_1 \times R_2 + R_2^2 = L_3^2 \tag{12}$$

$$R_2^2 - E_1 \times R_2 \times R_3 + R_3^2 = L_1^2 \tag{13}$$

$$R_3^2 - F_1 \times R_3 \times R_1 + R_1^2 = L_2^2 \tag{14}$$

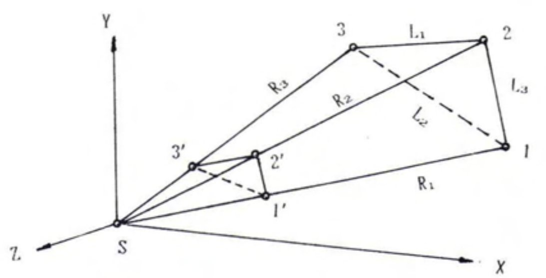

FIG. 1. The image pyramid (S is the exposure station; 1,2,3 are object control points).

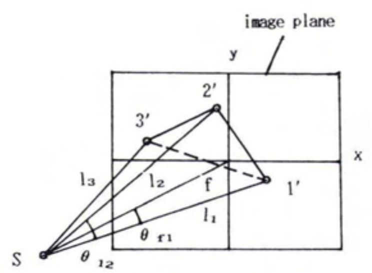

FIG. 2. The geometry of a image plane and exposure station.

This system yields eight solutions at most. Because only real positive solutions are needed here and each term of the unknowns is of second degree, there are at most four useful solutions to the system.

After dividing Equations 12 and 13 by Equation 14, and assuming that

$$K_{12} = (L_1/L_2)^2 \text{ and } K_{32} = (L_3/L_2)^2,$$

it follows that

$$R_2^2 - D_1 \times R_1 \times R_2 - K_{32} \times R_3^2 + K_{32} \times F_1 \times R_1 \times R_3 + R_1^2 \times (1 - K_{32}) = 0 \tag{15}$$

$$R_2^2 - E_1 \times R_2 \times R_3 - K_{12} \times R_1^2 + K_{12} \times F_1 \times R_1 \times R_3 + R_3^2 \times (1 - K_{12}) = 0 \tag{16}$$

Equations 15 and 16 are homogeneous, so there exists an arbitrary scale factor for the unknowns. Assume that $p = R_1$. Then

$$R_1/p = 1, \quad R_2/p = m, \quad R_3/p = n. \tag{17}$$

Substituting these relations into Equation 15, we have

$$m = (D_1 \pm \sqrt{D_1^2 - 4 \times (-K_{32} \times n^2 + K_{32} \times F_1 \times n + 1 - K_{32})})/2 \tag{18}$$

From Equation 16, we have

$$m = (E_1 \times n \pm \sqrt{E_1^2 \times n^2 - 4 \times (-K_{12} + K_{12} \times F_1 \times n + n^2 \times (1 - K_{12}))})/2 \tag{19}$$

Setting Equation 18 equal to Equation 19, one obtains

$$D_1 - E_1 \times n = \pm \sqrt{E_1^2 \times n^2 - 4 \times (-K_{12} + K_{12} \times F_1 \times n + n^2 \times (1 - K_{12}))}$$

$$\mp \sqrt{D_1^2 - 4 \times (-K_{32} \times n^2 + K_{32} \times F_1 \times n + 1 - K_{32})} \tag{20}$$

By taking square of Equation 20 two times, the square root operation is eliminated. After appropriate manipulation, it follows that

$$N_1 \times n^4 + N_2 \times n^3 + N_3 \times n^2 + N_4 \times n + N_5 = 0 \tag{21}$$

where

$$N_1 = (1 - K_{12})^2 + K_{32}^2 - 2 \times (1 - K_{12}) \times K_{32} - K_{32} \times E_1^2 + 4 \times K_{32} \times (1 - K_{12}) \tag{22}$$

$$N_2 = K_{32} \times F_1 \times (-2 - 2 \times K_{32} + 4 \times K_{12} + E_1^2) + K_{12} \times (E_1 \times D_1 + 2 \times F_1 - 2 \times K_{12} \times F_1) + E_1 \times D_1 \times (K_{32} - 1) \tag{23}$$

$$N_3 = K_{32} \times (K_{32} \times F_1^2 - F_1 \times D_1 \times E_1 - 4 \times K_{12} + 2 \times K_{32} - E_1^2 - 2 \times K_{12} \times F_1^2) + K_{12} \times (K_{12} \times F_1^2 + 2 \times K_{12} - F_1 \times E_1 \times D_1 - D_1^2) + D_1^2 - 2 + E_1^2 \tag{24}$$

$$N_4 = K_{32} \times (2 \times F_1 - 2 \times K_{32} \times F_1 + E_1 \times D_1 + 4 \times K_{12} \times F_1) + K_{12} \times (E_1 \times D_1 - 2 \times K_{12} \times F_1 + F_1 \times D_1^2 - 2 \times F_1) - D_1 \times E_1 \tag{25}$$

$$N_5 = K_{32} \times (K_{32} - 2 - 2 \times K_{12}) + K_{12} \times (K_{12} - D_1^2 + 2) + 1 \tag{26}$$

Assuming that

$$M_2 = N_2/N_1, \quad M_3 = N_3/N_1, \quad M_4 = N_4/N_1, \quad M_5 = N_5/N_1;$$

Equation 21 is changed into following:

$$n^4 + M_2 \times n^3 + M_3 \times n^2 + M_4 \times n + M_5 = 0 \tag{27}$$

Equation 27 is a typical algebraic equation of fourth degree and has a general solution. Although the method of solution is well-known, some practical expedients must be taken in order to obtain the solution (see Appendix). We can obtain four roots n_1, n_2, n_3, and n_4 of Equation 27. Among them, one to four roots will have real positive values which can be used to find the exposure station coordinates.

From Equation 17, we have

Introducing these values into Equation 14, we obtain

$$R_1 = p = L_2 / \sqrt{n^2 - F_1 \times n + 1} \tag{28}$$

$$R_3 = n \times p \tag{29}$$

Introducing R_1 into Equation 12, we obtain two solutions for R_2; that is,

$$R_2 = (D_1 \times R_1 \pm \sqrt{D_1^2 \times R_1^2 - 4 \times (R_1^2 - L_3^2)})/2 \tag{30}$$

Introducing R_3 into Equation 13, we obtain another two solutions of R_2:

$$R_2 = (E_1 \times R_3 \pm \sqrt{E_1^2 \times R_3^2 - 4 \times (R_3^2 - L_1^2)})/2 \tag{31}$$

If the values of R_2 from Equations 30 and 31 are equal (except for calculation error), then we select that value as the correct R_2. There must be one such R_2.

The exposure station corresponding to R_1, R_2, R_3 is given by the intersection point of three spheres with radius R_1, R_2, R_3 and centers at the three control points. The equations for three spheres may be written as follows:

$$(X_{L1} - X_s)^2 + (Y_{L1} - Y_s)^2 + (Z_{L1} - Z_s)^2 = R_1^2$$
$$(X_{L2} - X_s)^2 + (Y_{L2} - Y_s)^2 + (Z_{L2} - Z_s)^2 = R_2^2 \quad (32)$$
$$(X_{L3} - X_s)^2 + (Y_{L3} - Y_s)^2 + (Z_{L3} - Z_s)^2 = R_3^2$$

where X_{Li}, Y_{Li}, Z_{Li} are the coordinates of the control points in the ground system, and X_s, Y_s, Z_s are the coordinates of the exposure station.

After expanding, we have

$$X_s^2 + Y_s^2 + Z_s^2 - 2 \times X_{L1} \times X_s - 2\,Y_{L1} \times Y_s - 2\,Z_{L1} \times Z_s = R_1^2 - (X_{L1}^2 + Y_{L1}^2 + Z_{L1}^2) \quad (33)$$

$$X_s^2 + Y_s^2 + Z_s^2 - 2 \times X_{L2} \times X_s - 2\,Y_{L2} \times Y_s - 2\,Z_{L2} \times Z_s = R_2^2 - (X_{L2}^2 + Y_{L2}^2 + Z_{L2}^2) \quad (34)$$

$$X_s^2 + Y_s^2 + Z_s^2 - 2 \times X_{L3} \times X_s - 2\,Y_{L3} \times Y_s - 2\,Z_{L3} \times Z_s = R_3^2 - (X_{L3}^2 + Y_{L3}^2 + Z_{L3}^2) \quad (35)$$

Subtracting Equation 34 from 33, and Equation 35 from 34, we obtain

$$(X_{L2} - X_{L1})\,X_s + (Y_{L2} - Y_{L1})\,Y_s + (Z_{L2} - Z_{L1})\,Z_s = (R_1^2 - R_2^2 + (X_{L2}^2 + Y_{L2}^2 + Z_{L2}^2) - (X_{L1}^2 + Y_{L1}^2 + Z_{L1}^2))/2 \quad (36)$$

$$(X_{L3} - X_{L2})\,X_s + (Y_{L3} - Y_{L2})\,Y_s + (Z_{L3} - Z_{L2})\,Z_s = (R_2^2 - R_3^2 + (X_{L3}^2 + Y_{L3}^2 + Z_{L3}^2) - (X_{L2}^2 + Y_{L2}^2 + Z_{L2}^2))/2 \quad (37)$$

From the above two equations, two unknowns (e.g., X_s and Y_s) can be solved for in terms of the third unknown (e.g., Z_s); by introducing their values into Equations 33 to 35, two solutions of Z_s can be found. Then two sets of exposure station coordinates X_{s1}, Y_{s1}, Z_{s1} and X_{s2}, Y_{s2}, Z_{s2} are determined for each set of R_i.

From Figure 1, a mixed product can be made from vectors **12**, **13**, **1S**; that is,

$$T = \mathbf{1S} \cdot (\mathbf{12} \times \mathbf{13})$$

The two exposure stations give two values of T which have different signs. If T is positive, the exposure station is determined correctly. So from at most four sets of R_i, one to four solutions of the exposure stations are finally determined.

To sum up, the routine of finding the exposure stations is as follows:

(1) L_1, L_2, L_3 are computed from Equations 2 to 4 using the given coordinates of control points in the ground coordinate system.
(2) D_1, E_1, F_1 are computed from Equations 8 and 10 using the image cooordinates.
(3) M_2, M_3, M_4, M_5, the coefficients of the biquadratic polynomial, are calculated from Equations 22 to 27.
(4) Solving the biquadratic Equation 27, one to four real positive solutions of n are obtained.
(5) From each positive value of n, a set of R_1, R_2, R_3 is computed from Equations 28 to 31.
(6) From each set of R_1, R_2, R_3, the coordinates of exposure stations are computed from Equations 33 to 37. After computing the value of T, the correct exposure station coordinates are selected.

THE METHOD OF FINDING THE ROTATIONAL PARAMETERS

ROTATIONAL PARAMETERS USED IN THE COMPUTATION

Because the singularity problem exists in the computation of trigonometric functions, attempts have been made by photogrammetrists to construct the rotational matrix by using algebraic parameters (Thompson, 1959; Schut, 1958-59). However, these methods have not proven practical (Zeng, 1990).

Pope (1970) proposed four algebraic parameters d, a, b, c to construct the rotation matrix **R**. Hinsken (1988) derived a complete set of formulae to make the Pope parameters applicable in practical photogrammetric computation. Details of the Pope-Hinsken algorithm can be found in Hinsken (1988) and Zeng (1990).

$$\mathbf{R} = \begin{bmatrix} d^2 + a^2 - b^2 - c^2 & 2(ab + cd) & 2(ac - bd) \\ 2(ab - cd) & d^2 - a^2 + b^2 - c^2 & 2(bc + ad) \\ 2(ac + bd) & 2(bc - ad) & d^2 - a^2 - b^2 + c^2 \end{bmatrix} \quad (38)$$

where

$$d^2 + a^2 + b^2 + c^2 = 1 \quad (39)$$

It has been proven in practice that the Pope-Hinsken algorithm has no problem of singularity and readily converges (i.e., it has a large convergence radius).

To give a physical interpretation to these parameters, assume that

$$\mathbf{M} = [a\ b\ c]^T.$$

Then, we can get

$$\mathbf{R}\ \mathbf{M} = \mathbf{M}. \tag{40}$$

Hence, after transforming **M** by **R**, **M** remains unchanged. The physical meaning of a,b,c is shown in Figure 3. After the transformation of **R** is executed, the coordinate system is rotated by an angle γ around vector **M** (the rotation axis).

In view of Equation 39 and Figure 3, it follows that

$$|\mathbf{M}| = L_m = \sqrt{a^2 + b^2 + c^2} = \sqrt{1 - d^2}. \tag{41}$$

Assume that

$$a = b = c = S_o.$$

Vector **M** is written as follows:

$$\mathbf{M} = S_o\,\mathbf{i} + S_o\,\mathbf{j} + S_o\,\mathbf{k}$$
$$L_m = \sqrt{3}\,S_o \tag{42}$$

Vector **t**, perpendicular to vector **M**, can be expressed as follows:

$$\mathbf{t} = t_1\,\mathbf{i} + t_2\,\mathbf{j} + t_3\,\mathbf{k}. \tag{43}$$

From the condition of orthogonality, we have

$$\mathbf{t} \circ \mathbf{M} = 0.$$

Then, it follows that

$$t_1 + t_2 + t_3 = 0.$$

Assume, without loss of generality, that

$$t_1 = t_2 = t_o, \quad t_3 = -\,2 \times t_o$$

Let the angle between vector **t** and **R** transformed **t** be denoted by γ. Then

$$\frac{\mathbf{t} \circ \mathbf{R}\,\mathbf{t}}{|\mathbf{t}| \times |\mathbf{t}|} = \cos\gamma \tag{44}$$

where

$$|\mathbf{t}| = \sqrt{6} \times t_o$$

By expanding and manipulating Equation 44, it follows that

$$1 - 6\,S_o^2 = \cos\gamma$$

$$L_m = \sqrt{(1 - \cos\gamma)/2} = \sqrt{1 - d^2} \tag{45}$$

Thus, the rotation angle γ of the coordinate system has been expressed in terms of the rotation parameter d by Equation 45.

Finding the Rotation Axis

Assume that

al,bl — two unit vectors of control points with origin at exposure station,
ap,bp — two unit vectors of image points of corresponding control points,
g (X,Y,Z) — rotation axis of coordinate system from object space to image space.

The plane perpendicular to the plane formed by **ap** and **al** and bisecting the angle between **ap** and **al** is the plane *sm1g*. Using **bp,bl**, a similar plane *sm2g* can be formed. The intersection line of *sm1g* and *sm2g* gives the rotation axis **g** (Figure 4).

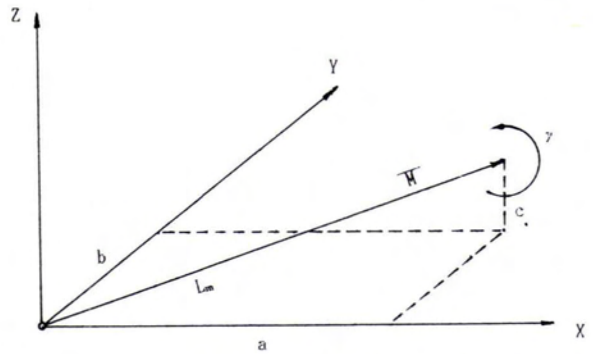

FIG. 3. The physical meaning of a,b,c.

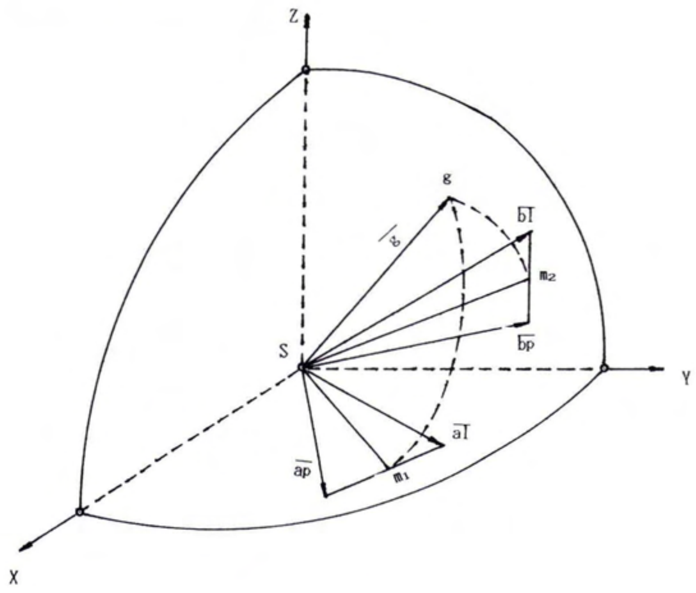

FIG. 4. Determination of the rotation axis.

Plane $sm1g$ is defined by

$$\mathbf{g} \times \mathbf{ap} - \mathbf{g} \times \mathbf{al} = 0 \tag{46}$$

Plane $sm2g$ is defined by

$$\mathbf{g} \times \mathbf{bp} - \mathbf{g} \times \mathbf{bl} = 0 \tag{47}$$

Using

$$\mathbf{ap} = [ap_x, ap_y, ap_z]^T, \quad \mathbf{bp} = [bp_x, bp_y, bp_z]^T,$$

Equations 46 and 47 are expanded into the following:

$$(ap_x - al_x)\,X + (ap_y - al_y)\,Y + (ap_z - al_z)\,Z = 0 \tag{48}$$

$$(bp_x - bl_x)\,X + (bp_y - bl_y)\,Y + (bp_z - bl_z)\,Z = 0 \tag{49}$$

Then the direction numbers of the rotation axis **g** are

$$\begin{aligned} p &= (ap_y - al_y)(bp_z - bl_z) - (ap_z - al_z)(bp_y - bl_y) \\ q &= (ap_z - al_z)(bp_x - bl_x) - (ap_x - al_x)(bp_z - bl_z) \\ r &= (ap_x - al_x)(bp_y - bl_y) - (ap_y - al_y)(bp_x - bl_x) \end{aligned} \tag{50}$$

FINDING THE ROTATION PARAMETERS *d, a, b, c*

The rotating angle of the coordinate system from object space to image space is γ; that is,

$$\cos\gamma = \frac{(\mathbf{g} \times \mathbf{ap}) \circ (\mathbf{g} \times \mathbf{al})}{|\mathbf{g} \times \mathbf{ap}| \times |\mathbf{g} \times \mathbf{al}|} \tag{51}$$

From Equation 45, it follows that

$$d = \sqrt{(1 + \cos\gamma)/2} \tag{52}$$

Because the direction numbers and coordinate components of the rotation axis **g** are proportional, then from Equation 39, it follows that

$$a = p\,\sqrt{(1 - d^2)/(p^2 + q^2 + r^2)} \tag{53}$$

$$b = q\,\sqrt{(1 - d^2)/(p^2 + q^2 + r^2)} \tag{54}$$

$$c = r\,\sqrt{(1 - d^2)/(p^2 + q^2 + r^2)} \tag{55}$$

In the above derivation, two object control points are enough to find d,a,b,c (see Figure 4); thus, the third point can be used for checking. Because there exist one to four solutions for the exposure station, this leads to one to four solutions for the rotation parameters as well.

If the fourth point exist, transforming the image coordinates of that point to object space by **R** (constructed by d, a, b, c or d, $-a$, $-b$, $-c$), we obtain a vector in the object space system. By checking whether the fourth control point is situated on the vector, a final unique solution for the exterior orientation parameters can be selected.

In order to perform the check, the direction numbers of the vector are used and the direction number in x-direction is defined as 1.0. In fact, two coordinates of the fourth point are enough for this checking.

To sum up, the routine for finding the rotational parameters is as follows:

(1) The vectors of image points are normalized (becoming unit vectors). Taking the exposure station as origin, the vectors of control points are constructed and normalized.
(2) From the vectors of two control points and their corresponding image vectors, the direction numbers p,q,r of the rotation axis are computed using Equation 50.
(3) The rotating angle γ is computed using Equation 51.
(4) d is computed using Equation 52.
(5) The rotational parameters a,b,c are computed using Equations 53 to 55.
(6) Using d,a,b,c, the rotation matrix **R** is constructed by using Equation 38. After transforming the image vector of the fourth point into object space by **R** and checking whether the fourth control point is situated on the vector, a final unique solution can be selected.

PRACTICAL TEST

A program has been written to realize this algorithm. One set of practical data measured at the inner test field of the Institute of Mine Surveying RWTH Aachen, Germany is used to illustrate the effectiveness of the algorithm.

The object coordinates (in m) are given in Table 1.

The image coordinates (in mm, focal length $f=63.874$ mm) are given in Table 2.

The coordinates of the exposure station are

$$X_s = 95.562 \text{ m}, Y_s = 117.443 \text{ m}, Z_s = 9.632 \text{ m}$$

The rotation parameters are

$$d = 0.385499, a = 0.376372, b = -0.589498, c = -0.601852$$

These parameters correspond to the angular parameters

$$\alpha = 127.558°, \beta = 101.379°, \gamma = .092°$$

By using points 1, 2, and 3, the results of the computation from this program are as follows:

$$n_1=\text{complex}, n_2=\text{complex}, n_3=1.458805, n_4=0.860057$$

Solution 1 gives

$$X_s=95.568 \text{ m}, Y_s=117.448 \text{ m}, Z_s=9.632 \text{ m}$$
$$d=0.385300, a=0.376255, b=-0.589595, c=-0.601958$$

Solution 2 gives

$$X_s=110.512 \text{ m}, Y_s=108.764 \text{ m}, Z_s=-4.362 \text{ m}$$
$$d=0.024307, a=0.531449, b=-0.845769, c=0.040556$$

Checking with the fourth point, the image coordinates are transformed into the ground system. The results for solutions 1 and 2 are shown in Table 3 and Table 4, respectively. After the direction numbers in the x-direction are defined as 1.0, we can compare whether the vectors of the fourth point computed from image point and control point are equal or not.

It is clear that solution 1 is the correct choice.

TABLE 1. THE OBJECT COORDINATES (IN M)

No.	X	Y	Z
1	107.9605	115.7181	12.0221
2	110.7004	106.7036	5.4821
3	106.2431	102.2492	8.9984
4	110.8310	112.8439	8.9997

TABLE 2. THE IMAGE COORDINATES (IN MM)

No.	X	Y
1	−19.460	14.218
2	11.814	−13.100
3	36.978	−1.188
4	−9.028	−1.165

TABLE 3. THE RESULTS OF SOLUTION 1

point No.	image			object point		
	X	Y	Z	X	Y	Z
1	1.000	−.139	.193	1.000	−.139	.193
2	1.000	−.710	−.274	1.000	−.710	−.274
3	1.000	−1.423	−.059	1.000	−1.423	−.059
4	1.000	−.301	−.041	1.000	−.301	−.041

TABLE 4. THE RESULTS OF SOLUTION 2

point No.	image			object point		
	X	Y	Z	X	Y	Z
1	1.000	−2.732	−6.432	1.000	−2.722	−6.414
2	1.000	−11.264	53.831	1.000	−11.130	53.228
3	1.000	1.521	−3.122	1.000	1.525	−3.128
4	1.000	−24.532	−150.725	1.000	12.915	42.292

THE MULTIPLE-VALUED PROBLEM OF SPACE RESECTION

Fischler and Bolles (1981) discussed the multiple-valued nature of space resection in detail. Figure 5 shows a tetrahedron with an equilateral triangle base ABC and with legs which are all equal. In leg SA there exists another point A′, from which the constructed triangle A′BC is equal to ABC. This gives the second solution for the exposure station. The same cases exist for two other legs. These demonstrate the four real solutions to the system of Equations 12 to 14. Fischler and Bolles (1981) have found a fourth point D (Figure 5) that moves to D′ on the line SD when A moves to A′. Thus, A,B,C,D give the same four solutions. In other words, four points in general position do not produce a unique solution. They concluded that: first, when four control points lie in a common plane (not containing the exposure station, and such that no more than two of the control points lie on any single line), they can produce a unique solution; and second, six (or more) control points in general position will always produce a unique solution. In our opinion, the first conclusion is really correct, but the second one is doubtful. Let's discuss this point in detail.

Assume that two solutions have the following results:

Exposure stations:

left station: X_{s1}, Y_{s1}, Z_{s1}

right station: X_{s2}, Y_{s2}, Z_{s2}

Rotation matrix:

From the left rotation parameters d_1, a_1, b_1, c_1, construct the following rotation matrix $\mathbf{T}_1$ by using Equation 38:

$$\mathbf{T}_1 = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad (56)$$

The corresponding data for the right rotation parameters are d_2, a_2, b_2, c_2 and $\mathbf{T}_2$:

$$\mathbf{T}_2 = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad (57)$$

We will show how many points can meet these two solutions. Can there be more than five?

Assume that the image point has image coordinates $x, y, z=-f$ where f is the principal distance. Then a point which meets the solutions must satisfy the coplanar condition of three vectors of the left image point, the right image point (they are the same point in the image plane but, after being transformed to different exposure stations, they have different vectors), and the base from the left exposure station to the right exposure station.

Assume that the base vector has three components: $B_x=X_{s2}-X_{s1}$, $B_y=Y_{s2}-Y_{s1}$, $B_z=Z_{s2}-Z_{s1}$. The image vectors have following components:

Left image vector:

$$\begin{aligned} x_1 &= l_{11} \times x + l_{12} \times y + l_{13} \times z \\ y_1 &= l_{21} \times x + l_{22} \times y + l_{23} \times z \\ z_1 &= l_{31} \times x + l_{32} \times y + l_{33} \times z \end{aligned} \quad (58)$$

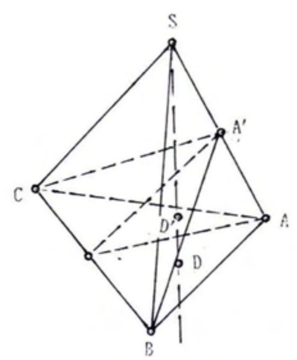

FIG. 5. The tetrahedron with an equilateral triangle base.

Right image vector:

$$\begin{aligned} x_2 &= r_{11} \times x + r_{12} \times y + r_{13} \times z \\ y_2 &= r_{21} \times x + r_{22} \times y + r_{23} \times z \\ z_2 &= r_{31} \times x + r_{32} \times y + r_{33} \times z \end{aligned} \quad (59)$$

The coplanar condition is as follows:

$$\begin{vmatrix} B_x & B_y & B_z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0 \quad (60)$$

Expanding Equation 60, we obtain an algebraic equation of second degree as follows:

$$N_1x^2 + N_2y^2 + N_3z^2 + N_4xy + N_5yz + N_6zx = 0 \quad (61)$$

Equation 61 represents a surface of second degree containing the exposure station. After substitution of $z = -f$ into Equation 61, we get

$$N_1x^2 + N_2y^2 + N_4xy + N_5'y + N_6'x + N_3' = 0 \quad (62)$$

Equation 62 represents a curve of second degree on the image plane.

This means that any image point which lies on this curve represents an object point which fits the two solutions. So there are an infinite number of points which can fit the two solutions and we can not draw a conclusion that six points in general position will always produce a unique solution.

To demonstrate this feature, we take the example shown in Figure 5. Three control points are vertices of an equilateral triangle. The coordinates of the control points are listed in Table 5.

The coordinates of exposure station are $X_s = 0$, $Y_s = 0$, $Z_s = 70$ m. The coordinate system of the image is parallel to the ground system. The program gives four exposure stations which have the coordinates (Figure 6) listed in Table 6.

For stations S_1 and S_4, Equation 61 gives the following:

$$40.6989\, xy - 8.3919\, xz = 0$$

or

$$x(y - 0.20619\, z) = 0 \quad (63)$$

When the focal length $f = 0.07$ m and $z = -f$, namely on the image plane, Equation 63 represents two straight lines: $x = 0$ and $y = -0.014433$ m. They are presented in the image plane as a bisector of angle A′ and one side B′C′ of the triangle (Figure 7).

From stations S_1 and S_2, Equation 61 gives the following:

$$9.9388x^2 - 9.9388y^2 + 11.4752xy - 30.0005yz - 17.3217zx = 0$$

or

$$(x + 1.73205\, y)\,(x - 0.5773\, y + 0.01667) = 0 \quad (64)$$

Equation 64 degenerates into two straight lines on the image plane. They are a bisector of angle C′ and side A′B′. The similar case is valid for stations S_1 and S_3.

TABLE 5. THE COORDINATES OF THREE CONTROL POINTS A,B,C

No.	X (m)	Y (m)	Z (m)
B	−25.0	−14.4337	0
C	25.0	−14.4337	0
A	0	28.8675	0

TABLE 6. THE COORDINATES OF FOUR EXPOSURE STATIONS

	Xs	Ys	Zs
S_1	0	0	70
S_2	39.099	−22.575	39.476
S_3	−39.099	−22.575	39.477
S_4	0	45.148	39.476

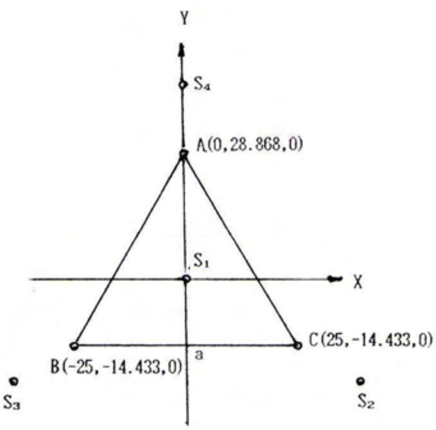

FIG. 6. The four exposure stations obtained from the tetrahedron in Figure 5.

From S_2 and S_4, Equation 61 gives the following:

$$-41.3428x^2 + 41.3427y^2 + 47.7404xy + 17.0497yz - 9.8437zx = 0$$

or

$$(x - 1.73205\, y)\,(x - 0.5773\, y - 0.01667) = 0 \tag{65}$$

Equation 65 degenerates to lines A′C′ and B′b′.

Hence, we can draw an important conclusion: for the space resection shown in Figures 5 and 6, in accordance with each image point positioned on the sides and bisectors (including the extension parts) of the triangle A′B′C′ (or in common, on the line represented by Equation 62), there exists a certain control point calculated by forward intersection from the two exposure stations (the two solutions). That point does not provide more information to do the space resection than did the three original points.

To illustrate the real positions of these control points, we pass a plane through a, S_1, A, S_4 (Figure 8). It is clear that a,S_1,A, S_4 are positioned on a common circle (curve). Any control points lying on this curve do not provide more information to do the space resection. We call it the critical curve. The sides themselves of the triangle ABC are locations of such control points as well.

Taking the data from RWTH Aachen as a practical example, Equation 62 gives the following:

$$x^2 + 6.246411y^2 + 158.4873xy - 67.03379x - 7.57918y + 302.4823 = 0 \tag{66}$$

Equation 66 is a hyperbola whose graph is shown in Figure 9. The fourth image point does not lie on this hyperbola.

The program contains a subroutine which tests the fourth image point as to whether it lies on the curve presented by Equation 62. If so, it tests further by forward intersection to determine if it lies on the critical surface.

CONCLUSION

The space resection utilizing three control points can be solved successfully with the algorithm proposed in this paper without iteration and initial information regarding the exterior orientation. The space resection produces one to four applicable solutions. If there is a fourth point and it does not lie on the critical curve, they can produce a unique solution.

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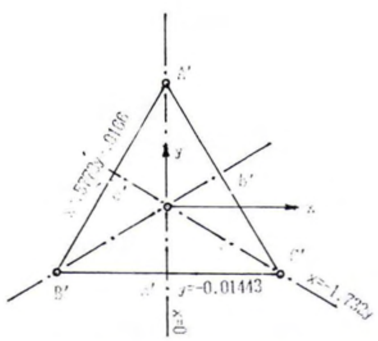

FIG. 7. The graph of Equation 62 on the image plane for the Case of Figure 5.

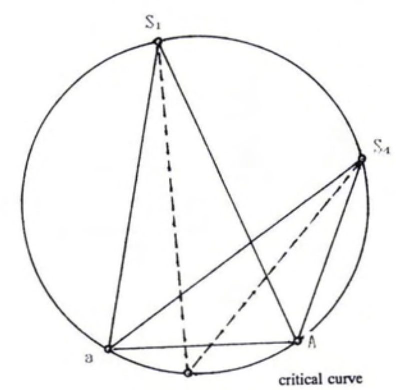

FIG. 8. a, A, S1, S4 are positioned on a critical curve.

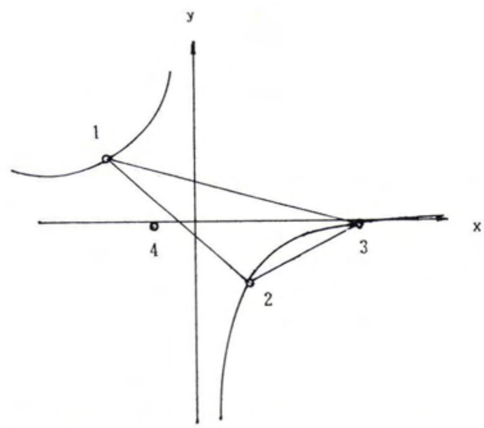

FIG. 9. Practical example from RWTH Aachen.

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APPENDIX

The general solution of an algebraic equation of fourth degree is

$$X^4 + a_1 \times X^3 + a_2 \times X^2 + a_3 \times X + a_4 = 0 \tag{A1}$$

Let

$$X = Y - a_1/4 \tag{A2}$$

Then Equation A1 becomes

$$Y^4 + b_1 \times Y^2 + b_2 \times Y + b_3 = 0 \tag{A3}$$

If $b_2 = 0$, then four solutions can be obtained from following equation:

$$Y^2 = (-b_1 \pm \sqrt{b_1^2 - 4\, b_3})/2 \tag{A4}$$

Otherwise, a real solution of the following cubic equation is found; that is,

$$U^3 - b_1 \times U^2 - 4 \times b_3 \times U + (4 \times b_1 \times b_3 - b_2^2) = 0 \tag{A5}$$

Assume that

$$U = V + b_1/3 \tag{A6}$$

Then Equation A5 becomes

$$V^3 + P \times V + q = 0 \tag{A7}$$

If

$$(q/2)^2 + (p/3)^3 = K > 0,$$

one real solution, V_1 is

$$V_1 = \sqrt[3]{-q/2 + \sqrt{(q/2)^2 + (p/3)^3}} + \sqrt[3]{-q/2 - \sqrt{(q/2)^2 + (p/3)^3}} \tag{A8}$$

If

$$(q/2)^2 + (p/3)^3 = K < 0,$$

one real solution V_1 is

$$V_1 = G \times \sqrt[3]{-q/2 + \sqrt{(q/2)^2 + (p/3)^3}} + G^2 \times \sqrt[3]{-q/2 - \sqrt{(q/2)^2 + (p/3)^3}} \tag{A9}$$

where

$$G = (-1 + \mathrm{i} \times \sqrt{3})/2, \quad G^2 = (-1 - \mathrm{i} \times \sqrt{3})/2$$

$$\mathrm{i} = \sqrt{-1}$$

By derivation, V_1 is obtained as

$$V_1 = 2 \times \sqrt{-p/3} \times \cos(pi - \tan^{-1}\sqrt{3} + w/3) \tag{A10}$$

where

$$pi = 3.14159$$

If $q > 0$

$$w = pi - \tan^{-1}(\sqrt{-K}/(q/2)) \tag{A11}$$

If $q < 0$

$$w = \tan^{-1}(\sqrt{-K}/(-q/2)) \tag{A12}$$

If $q = 0$

$$w = pi/2 \tag{A13}$$

Substitution of V_1 into Equation A6 yields a solution U_1 of Equation A5.

Then four solutions Y_1, Y_2, Y_3, Y_4 of Equation A3 are obtained by the following equivalent equations:

$$Y^2 + \sqrt{U_1 - b_1}\, Y + (U_1/2 - b_2/(2 \times \sqrt{U_1 - b_1})) = 0$$

$$Y^2 - \sqrt{U_1 - b_1}\, Y + (U_1/2 + b_2/(2 \times \sqrt{U_1 - b_1})) = 0 \tag{14}$$

Substitution of Y_1, Y_2, Y_3, Y_4 into Equation 2A yields the four solutions X_1, X_2, X_3, X_4 of Equation A1.