Initial Approximations for the Three-Dimensional Conformal Coordinate Transformation

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Abstract

The three-dimensional conformal coordinate transformation has many applications. In terrestrial photogrammetry it can be used to transform arbitrary stereomodel coordinates to object space systems. Determination of initial approximations in such situations is typically not as straightforward as in the aerial case. A method is presented which allows a direct solution of initial approximations given any arbitrary angular orientation and three-dimensional control.

Introduction

The three-dimensional conformal (seven-parameter) transformation is commonly used when transforming coordinates from a stereomodel into an object space system, or to link adjacent stereomodels to form a strip. It uses seven parameters: scale; three rotations (omega, phi, and kappa); and translations in *x*, *y*, and *z*. When dealing with nominally vertical photography, determination of initial approximations for the non-linear, iterative solution is straightforward because the values of omega and phi can be assumed to be zero (Moffitt and Mikhail, 1980). However, in close-range photogrammetry, the values of the three rotation angles can take on any value, and assumption of zero values generally leads to a divergent solution. To overcome this difficulty, a method has been devised which enables accurate determination of initial approximations for all parameters, thus ensuring convergence of the iterative solution.

Three-Dimensional Rotations

The angular attitude of three-dimensional Cartesian coordinate axes in one system relative to another can be specified by three independent parameters. Two common sets of parameters are tilt, swing, azimuth and omega, phi, kappa. A discussion and derivation of the equations for these parameters can be found in Wolf (1983). From either of these two systems, a three-dimensional rotation matrix can be derived. The rotation matrix format is given in Equation 1: i.e.,

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{1}$$

The definitions of the elements of the rotation matrix in terms of tilt (t), swing (s), azimuth (α) and omega (ω), phi (φ), kappa (κ) are given in Equations 2 and 3, respectively: i.e.,

$$\begin{aligned}
m_{11} &= -\cos(\alpha)\cos(s) - \sin(\alpha)\cos(t)\sin(s) \\
m_{12} &= \sin(\alpha)\cos(s) - \cos(\alpha)\cos(t)\sin(s) \\
m_{13} &= -\sin(t)\sin(s) \\
m_{21} &= \cos(\alpha)\sin(s) - \sin(\alpha)\cos(t)\cos(s) \\
m_{22} &= -\sin(\alpha)\sin(s) - \cos(\alpha)\cos(t)\cos(s) \\
m_{23} &= -\sin(t)\cos(s) \\
m_{31} &= -\sin(\alpha)\sin(t) \\
m_{32} &= -\cos(\alpha)\sin(t) \\
m_{33} &= \cos(t)
\end{aligned} \tag{2}$$

$$\begin{aligned}
m_{11} &= \cos(\varphi)\cos(\kappa) \\
m_{12} &= \sin(\omega)\sin(\varphi)\cos(\kappa) + \cos(\omega)\sin(\kappa) \\
m_{13} &= -\cos(\omega)\sin(\varphi)\cos(\kappa) + \sin(\omega)\sin(\kappa) \\
m_{21} &= -\cos(\varphi)\sin(\kappa) \\
m_{22} &= -\sin(\omega)\sin(\varphi)\sin(\kappa) + \cos(\omega)\cos(\kappa) \\
m_{23} &= \cos(\omega)\sin(\varphi)\sin(\kappa) + \sin(\omega)\cos(\kappa) \\
m_{31} &= \sin(\varphi) \\
m_{32} &= -\sin(\omega)\cos(\varphi) \\
M_{33} &= \cos(\omega)\cos(\varphi)
\end{aligned} \tag{3}$$

For vertical aerial photography, the values of tilt or omega and phi are nominally zero. However, for close-range applications, the camera axis can be pointed upward from the horizon, causing one or more of these angles to be greater than 90 degrees. In order to analyze rotations involving such large values, numerical ranges encompassing each of these angular parameters must first be defined. Convenient ranges of values for these parameters are as follows:

tilt:	(0° to 180°)	omega:	(−180° to 180°)
swing:	(−180° to 180°)	phi:	(−90° to 90°)
azimuth:	(−180° to 180°)	kappa:	(−180° to 180°)

These ranges are chosen to fulfill two purposes. First, they enable all possible angular orientations between two sets of three-dimensional coordinate axes to be uniquely defined, thus circumventing the duality problem discussed by Shih (1990). Second, they enable a straightforward conversion between the tilt, swing, azimuth system and the omega, phi, kappa system. Note that the ranges shown for swing, azimuth, and kappa are not the typical 0° to 360°. This is done for convenience within the program.

With the above range definitions, converting between the two

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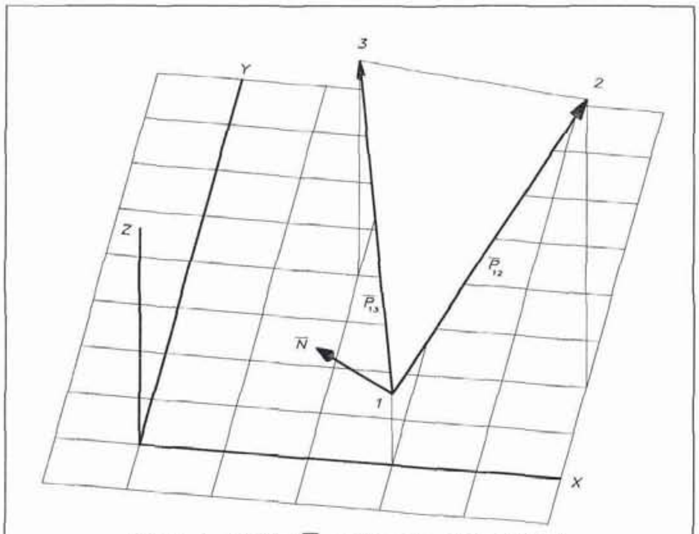

Figure 1. Vector $\overline{\mathbf{N}}$, normal to plane 1-2-3.

systems is elementary. It does require the use of an inverse tangent function which computes the full-circle range from $-180°$ to $180°$, such as the "atan2" function in FORTRAN or C. To convert from either system to the other, first the rotation matrix (or at least the necessary terms within it) is formed from the angles in the given system. Then, if the angles of tilt, swing, and azimuth are required, they can be computed by Equations 4, 5, and 6: i.e.,

$$\text{tilt} = \cos^{-1}(m_{33}) \tag{4}$$

$$\text{swing} = \tan^{-1}(-m_{13}/-m_{23}) \tag{5}$$

$$\text{azimuth} = \tan^{-1}(-m_{31}/-m_{32}) \tag{6}$$

Alternatively, given a rotation matrix, the values for omega, phi, and kappa can be computed by Equations 7, 8, and 9: i.e.,

$$\text{omega} = \tan^{-1}(-m_{32}/m_{33}) \tag{7}$$

$$\text{phi} = \sin^{-1}(m_{31}) \tag{8}$$

$$\text{kappa} = \tan^{-1}(-m_{21}/m_{11}) \tag{9}$$

The full-circle inverse tangent function (i.e., atan2) must be used with the indicated numerators and denominators, complete with leading minus signs where indicated, in order to obtain the proper quadrant. Other difficulties which could arise are determination of swing and azimuth when tilt is zero and determination of omega and kappa when phi is 90 degrees. In these cases, both the numerator and denominator are zero in Equations 5, 6, 7, and 9, giving undefined results from the inverse tangent function. Under these circumstances, Equations 10 and 11 or 12 and 13 can be applied: i.e.,

If tilt $=0°$:

$$\text{azimuth} = 0° \tag{10}$$

$$\text{swing} = \tan^{-1}(-m_{12}/-m_{22}) \tag{11}$$

If phi $= 90°$:

$$\text{omega} = 0° \tag{12}$$

$$\text{kappa} = \tan^{-1}(m_{12}/m_{22}) \tag{13}$$

Given the above treatment of the situation where tilt is zero, the usual problem of swing and azimuth being undefined is avoided. Visualization of what these angles mean can be difficult; however, mathematically it all works out.

Vectors and Planes

Part of the procedure for computing initial approximations for the rotations involves determination of the normal vector to a plane. The fundamentals are briefly presented here. Given three points (1, 2, and 3) in three-dimensional space, a vector (see Figure 1) which is normal to the plane defined by these points can be computed by use of the cross-product given in Equation 14. This cross-product can be computed using determinants (Thomas and Finney, 1982), as illustrated by Equation 15: i.e.,

$$\overline{\mathbf{N}} = \overline{\mathbf{P}}_{12} \times \overline{\mathbf{P}}_{13} \tag{14}$$

$$\overline{\mathbf{N}} = \begin{vmatrix} \bar{\mathbf{i}} & \bar{\mathbf{j}} & \bar{\mathbf{k}} \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} \tag{15}$$

Thus, Equation 16 shows the form of the normal vector, in terms of unit vectors: $\bar{\mathbf{i}}$, $\bar{\mathbf{j}}$, $\bar{\mathbf{k}}$: i.e.,

$$\overline{\mathbf{N}} = a \cdot \bar{\mathbf{i}} + b \cdot \bar{\mathbf{j}} + c \cdot \bar{\mathbf{k}} \tag{16}$$

where

$$a = (y_2-y_1)(z_3-z_1) - (y_3-y_1)(z_2-z_1)$$
$$b = (x_3-x_1)(z_2-z_1) - (x_2-x_1)(z_3-z_1)$$
$$c = (x_2-x_1)(y_3-y_1) - (x_3-x_1)(y_2-y_1)$$

In developing a procedure for determining the angular relationship between the two systems, it was decided to express the attitude of the normal vector (as if it was pointing along the negative z axis) in terms of tilt and azimuth. Figure 2 shows the normal vector placed at the origin for the purpose of determining these two angular parameters. Tilt is determined by calculating the elevation angle above the x-y plane and adding 90° to the result. From Figure 2, the expression for tilt, given in Equation 17, was obtained. The azimuth of the normal vector is equal to the azimuth of the projection of $\overline{\mathbf{N}}$ onto the x-y plane, and is given in Equation 18: i.e.,

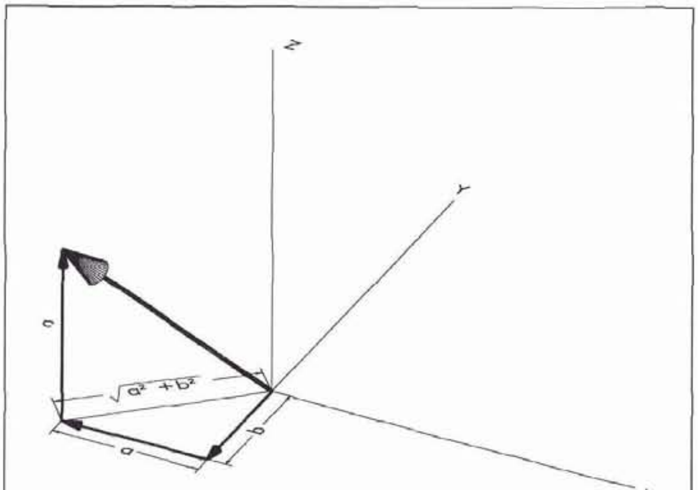

Figure 2. Normal vector, $\overline{\mathbf{N}}$, at origin.

$$\text{tilt} = \tan^{-1}\left(\frac{c}{\sqrt{a^2 + b^2}}\right) + 90^\circ \tag{17}$$

$$\text{azimuth} = \tan^{-1}\left(\frac{a}{b}\right) \tag{18}$$

where a, b, and c are the coefficients of $\bar{\mathbf{i}}$, $\bar{\mathbf{j}}$, and $\bar{\mathbf{k}}$, respectively.

In Equation 18, it is again necessary to employ the full-circle inverse tangent function (i.e., atan2) in order to obtain the proper quadrant.

Three-Dimensional Conformal Coordinate Transformation

A description and derivation of the three-dimensional conformal coordinate transformation can be found in Wolf (1983). Only a summary of the equations involved is presented here.

The seven unknown parameters for this transformation are

- s - *scale factor;*
- ω, φ, κ - *rotations about the* x, y, z axes, respectively; and
- T_x, T_y, T_z - translations of the origin in the x, y, z directions.

The form of the transformation is given in Equations 19, 20, and 21: i.e.,

$$X_c = s \cdot (m_{11}X_a + m_{21}Y_a + m_{31}Z_a) + T_x \tag{19}$$

$$Y_c = s \cdot (m_{12}X_a + m_{22}Y_a + m_{32}Z_a) + T_y \tag{20}$$

$$Z_c = s \cdot (m_{13}X_a + m_{23}Y_a + m_{33}Z_a) + T_z \tag{21}$$

In these equations, the m values are elements of the rotation matrix, previously specified as functions of ω, φ, and κ (Equation 3). X_c, Y_c, Z_c and X_a, Y_a, Z_a are coordinates of a point in the control and arbitrary systems, respectively. The equations are non-linear in terms of s, ω, φ, and κ, requiring an iterative least-squares solution.

In order for the least-squares solution to be achieved, it is necessary for coordinates of common points to be "known" (or measured) in both systems. These common points may consist of various combinations of horizontal and vertical control. In a practical case, all three coordinates (x, y, and z) will have been determined in the arbitrary (model) system for each common point. The procedure for determining initial approximations (to be described next) requires that full three-dimensional coordinates be known for at least three points in *both* systems. It is recognized that this constraint (not allowing separate horizontal and vertical control) is a limitation of the method. However, when this method is used in conjunction with close-range applications, this constraint can be minimized by determining three-dimensional coordinates of the exposure stations and using them as two of the required control points. This leaves only one additional three-dimensional control point which must be established in object space. If additional common points are available, various combinations of three points could be used, and the results averaged. This would lead to a more refined set of values and, perhaps more importantly, indicate the presence of blunders in the observations.

Initial Approximation for Scale

Computation of the scale factor is straightforward. Select a representative pair of points and calculate a ratio of computed lengths of the line in the two systems. In the usual sense of the transformation (from arbitrary to control coordinates), this scale factor can be calculated by Equation 22: i.e.,

$$\text{scale} = \frac{\text{distance in the control system}}{\text{distance in the arbitrary system}} \tag{22}$$

Assuming there are no blunders and the representative line is sufficiently long, an accurate approximation for the scale factor can be determined this way.

Rotations of Omega, Phi, and Kappa

Determination of accurate approximations for the three angular rotations is the crux of the method. This process shall be presented in a sequence of seven steps.

(1) Select the Three Geometrically Strongest Points

In order to obtain sufficient geometric strength, three points having a widely distributed base must be selected. This is done to reduce the likelihood of choosing three points which are nearly collinear. An analogy can be found in linking stereomodels during strip formation in semi-analytical aerotriangulation, where the projection centers are used as common points in order to provide additional geometric strength. The choice of three points will be based on the points which form a triangle having the largest altitude, where altitude is defined as the perpendicular distance from the longest side, to the point not on that side. This determination can be made in either the control or arbitrary system if no blunders exist. Figure 3 illustrates how this altitude is defined. In this figure the longest side is labeled (a) and, because the specified altitude is perpendicular to this side, it must be internal to the triangle. A formula for the square of the altitude is given by Equation 23: i.e.,

$$h^2 = b^2 - \left(\frac{a^2 + b^2 - c^2}{2 \cdot a}\right)^2 \tag{23}$$

If more than three control points are available, the value of h^2 is computed for all combinations of three points and the set giving the largest is chosen.

(2) Compute the Normal Vectors at a Common Point in both Systems

Computation of the normal vectors requires the application of Equation 16 at the common point in both systems. This will result in two sets of a, b, and c coefficients. One set is based on the arbitrary coordinates and the other is based on the control coordinates.

(3) Determine Tilt and Azimuth for the Normal in each System

Tilt and azimuth of the normal vector in both systems are based on Equations 17 and 18.

(4) Perform an Initial Rotation of Points in both Systems Using Corresponding Values of Tilt and Azimuth

This rotation will cause the plane defined by the three points to be parallel to the x-y plane in both systems. It is accomplished by computing a rotation matrix for each system, based on their respective values for tilt and azimuth, with

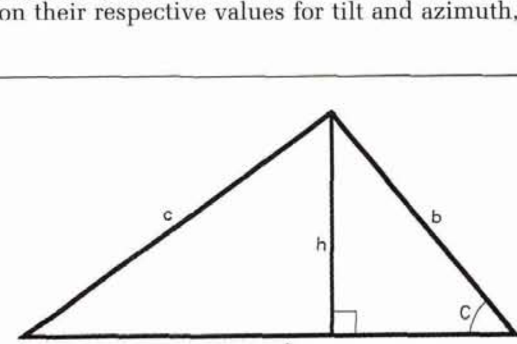

Figure 3. Altitude from longest side.

swing set equal to 0° (see Equation 2). Then this rotation matrix is applied to two of the three points in each system, resulting in transformed points having equal z coordinates. This is accomplished as illustrated by Equations 24, 25, and 26: i.e.,

$$x' = m_{11}x + m_{12}y + m_{13}z \tag{24}$$

$$y' = m_{21}x + m_{22}y + m_{23}z \tag{25}$$

$$z' = m_{31}x + m_{32}y + m_{33}z \tag{26}$$

These equations are applied a total of four times, once for each of the two points in the control system and once for each of the same two points in the arbitrary system. For the first two applications, the rotation matrix for the control system is used along with *x,y,z* control coordinates for the two points. For the last two applications, the rotation matrix for the arbitrary system is used along with corresponding *x,y,z* arbitrary coordinates.

The resulting prime coordinates define horizontal lines in both systems and require application of a swing value to a line in one system to give it the same direction as the line in the other system. Equation 26 is shown for clarity; however, it is unnecessary to compute the z' coordinates because they are not used in subsequent calculations.

(5) Determine Swing by Difference in Azimuths for the Common Line

Using the transformed coordinates of the two points as determined in Step 4, compute the azimuth of the connecting line in each system. The swing required to align the arbitrary system with the control system is then computed by Equation 27: i.e.,

$$\text{swing} = \text{azimuth(control)} - \text{azimuth(aribtrary)} \tag{27}$$

(6) Combine the Two Tilts, Two Azimuths, and One Swing Into One Overall Rotation Matrix

Rotation matrix $\mathbf{M}_1$ is formed using tilt and azimuth of the normal vector in the arbitrary system, in conjunction with the swing value determined in Step 5. Rotation matrix $\mathbf{M}_2$ is formed using only the tilt and azimuth of the normal vector in the control system (i.e., $\mathbf{M}_2$ remains as calculated in Step 4). Equation 28 shows how a point's coordinates in the two systems can subsequently be related: i.e.,

$$\mathbf{M}_1 \cdot \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} = \mathbf{M}_2 \cdot \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \tag{28}$$

The "equals" sign in this equation is not technically appropriate. Its meaning in this context is "corresponds to" or "is in a similar location as."

The rotation matrices are then combined into a single rotation matrix **M**, as shown in Equation 29, where the "equals" signs have the same meaning as in Equation 28: i.e.,

$$\begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} = \mathbf{M}_1^{\mathrm{T}} \cdot \mathbf{M}_2 \cdot \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \mathbf{M} \cdot \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} \tag{29}$$

(7) From Rotation Matrix M, Compute Values for Omega, Phi, and Kappa

This is accomplished by applying Equations 7, 8, and 9 or possibly 12 and 13.

Translations in X, Y, and Z

Because the transformation equations are linear in terms of T_x, T_y, and T_z, initial approximations are not required. However, if their values were desired, Equations 19, 20, and 21 could be rearranged so that the translation terms are isolated on one side of the equation. Computation of the translations can then be performed using a common point.

TABLE 1. DATA FOR EXAMPLE TRANSFORMATION.

Point	X-control (m)	Y-control (m)	Z-control (m)	x-arb (mm)	y-arb (mm)	z-arb (mm)
1	730412.363	83091.394	141.244	51.15	−23.62	−71.58
2	730576.273	83155.275	146.276	3.30	22.81	−42.89
3	730409.480	83277.516	143.536	20.44	20.50	−126.39
4	730604.274	83109.509	150.266	3.47	16.11	21.70

Example

To illustrate the method, the following example is given. Table 1 lists the coordinates of the common points in both systems.

Scale

The longest line from the given set of four points is line 3-4. After this has been determined, the scale computation involves the application of Equation 22 as illustrated below:

$$\text{scale} = \frac{\text{length of 3-4 (control)}}{\text{length of 3-4 (arbitrary)}} = \frac{257.325}{106.147} = 2.4242$$

Rotations

(1) Using every combination of three points from the total of four, lengths of the sides of a triangle are computed, along with the altitude. Table 2 summarizes these computations for each of the four combinations. The columns headed SIDE a, SIDE b, and SIDE c show the lengths of the three sides of the triangle being considered, with SIDE a being selected as the longest. Column *h* gives the computed altitude for the triangle. Note that points 1, 2, and 3 gave the strongest triangle (largest *h*, marked by *) and are therefore used for the rotation calculations.

(2) Computation of the normal vectors in both systems results from application of Equation 16 to the two sets of coordinates for points 1, 2, and 3. Table 3 shows a summary of the computation for both normal vectors at point 1. The columns COEFF a, COEFF b, and COEFF c show the computed normal vector coefficients.

(3) Application of Equations 17 and 18 to the coefficients from Table 3 yields the listed tilt and azimuth values.

(4) Using the tilt and azimuth values from Table 3, the following rotation matrices were calculated:

Arbitrary rotation matrix:

$$\begin{bmatrix} 0.676841792 & -0.736128514 & 0.000000000 \\ 0.096605956 & 0.088825453 & -0.991351264 \\ 0.729761933 & 0.670987965 & 0.131235178 \end{bmatrix}$$

Control rotation matrix:

$$\begin{bmatrix} 0.442772214 & -0.896634132 & 0.000000000 \\ -0.896264753 & -0.442589809 & -0.028701105 \\ 0.025734390 & 0.012708052 & -0.999588038 \end{bmatrix} = \mathbf{M}_2$$

TABLE 2. SUMMARY OF TRIANGLE COMPUTATION.

Points	SIDE a	SIDE b	SIDE c	h
1 2 3	206.810	186.158	175.990	148.465*
1 2 4	192.975	53.801	175.990	48.281
1 3 4	257.325	192.975	186.158	139.168
2 3 4	257.325	53.801	206.810	16.582

Table 3. Summary of Normal Vector Computation.

SYSTEM	COEFF a	COEFF b	COEFF c	Tilt	Azimuth
arbitrary	−3810.6311	−3503.7284	−685.276	82.4590°	−132.5973°
control	−790.1507	−390.1890	30691.426	178.3553°	−116.2809°

Table 4. Rotated Coordinates.

SYSTEM	x'_1	y'_1	x'_2	y'_2
Arbitrary	52.00781	73.80426	−14.55751	44.86396
Control	248903.72	−691422.31	248919.02	−691597.64

These rotation matrices are then applied, using Equations 24 and 25, to the corresponding coordinates of points 1 and 2 (from Table 1) to give the values listed in Table 4.

(5) Using coordinates of the endpoints of line 1-2 listed in Table 4, the following azimuths are computed:

$$\text{azimuth(arbitrary)} = -113.4977^\circ$$
$$\text{azimuth(control)} = 175.0136^\circ$$

Equation 27 is then applied to these azimuth values to yield the following value for swing:

$$\text{swing} = 288.5113^\circ$$

(6) Using the value for swing computed in Step 5 and the values for tilt and azimuth in the arbitrary system (from Table 3) computed in Step 3, the following rotation matrix, $\mathbf{M}_1$, is computed:

Arbitrary rotation matrix with swing:

$$\mathbf{M}_1 = \begin{bmatrix} 0.123284530 & -0.317944953 & 0.940059536 \\ 0.672494198 & -0.669840397 & -0.314746560 \\ 0.729761933 & 0.670987965 & 0.131235178 \end{bmatrix}$$

When matrices $\mathbf{M}_1^T$ and $\mathbf{M}_2$ are multiplied, the following rotation matrix, $\mathbf{M}$, is computed:

Overall rotation matrix:

$$\mathbf{M} = \begin{bmatrix} -0.529365903 & -0.398906344 & -0.748762625 \\ 0.476844613 & 0.590071780 & -0.651486385 \\ 0.701705747 & -0.701918103 & -0.122147540 \end{bmatrix}$$

(7) Using Equations 7, 8, and 9 results in the following values for omega, phi, and kappa:

$$\omega = 99.8717^\circ$$
$$\varphi = 44.5640^\circ$$
$$\kappa = -137.9880^\circ$$

Using these initial approximations, a three-dimensional conformal coordinate transformation was performed, giving the following set of corrections for the first iteration:

$$\Delta s = 0.0002$$
$$\Delta\omega = 0.0021^\circ$$
$$\Delta\varphi = 0.0063^\circ$$
$$\Delta\kappa = -0.0027^\circ$$

Conclusions

The foregoing method of computing initial approximations for a three-dimensional conformal coordinate transformation has been extensively tested and found to be successful in all cases, under the assumptions of no blunders and strictly three-dimensional control points. It is particularly useful in close-range photogrammetric applications where the unusual rotations can be difficult to predict or visualize.

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