Testing Camera Calibration with Constraints

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Abstract

Known geometric relationships between points, lines, and planes on a three-dimensional (3D) object can be used as constraints in the camera calibration process to improve the accuracy of estimates. Ten constraints have been constructed and tested numerically. The accuracy improvement has been evaluated theoretically and empirically by comparing the variances and traces of the covariance matrix of the 3D points obtained from the camera calibration without constraints with those derived from the process with constraints. It is not uncommon to observe an order of magnitude improvement in the variances of the estimates for problems having multiple constraints. Statistical tests comparing the theoretical prediction with the empirical observed data validate the approach.

Introduction

Camera calibration is a traditional topic in photogrammetry and computer vision. Originally, the camera calibration technique was used for determining the interior orientation elements (focal length, principal point, fidicial center, and radial distortion) (Karara, 1989). This concept has been generalized by considering the exterior orientation elements (rotation and translation) and the three-dimensional (3D) points as parameters to be estimated in the process. Because the parameters of interior orientation and some additional parameters are computed for every adjustment, as long as the photogrammetrist checks to see that the magnitude of these parameters is "reasonable," no lasting importance need be attached to them. It is usually the results for the object point coordinates and their error estimates that are of concern (Karara, 1989). The importance of camera calibration is not just for making accurate 3D measurement, but also for helping a 3D model-based vision system to model the performance or capability of any particular sensing strategy (Tsai, 1989). Features in an image — points, edges (or lines), and areas — are useful geometric information that can be employed for the camera calibration (Tommaselli and Tozzi, 1996; Karara, 1989; Echigo, 1990; Liu *et al.*, 1990). Some of these features may correspond to three-dimensional entities having a known relation between them. The known relation can be used as a constraint to improve the accuracy of the estimated 3D points. The major contributions of this paper are (1) the use of ten constraints which include the basic geometric relationships between points, lines, and planes on a 3D object; (2) the development of a method for the evaluation of the efficiency of the constraints; and (3) the proper handling of the "zero determinant" problem encountered in propagating the variances using the method discussed by Haralick (1993).

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Camera Calibration with Constraints

The Camera Calibration with Constraints performs a simultaneous estimation of the parameters of multiple cameras and some locations of the 3D passpoints under some specific constraints (Thornton, 1997; Huang and Haralick, 1997). For each image, there are 11 camera parameters consisting of the focal length, coordinates of the principal point, coordinates of the perspective center, the rotation angles, and a scale factor. Three-dimensional points whose coordinates are unknown are called passpoints. Three-dimensional points with known coordinates are called control points. Their coordinates are fixed in the camera calibration process. Two-dimensional (2D) points are the image points corresponding to the 3D points. The coordinates of the 2D points are the noisy observed values. Let $\mathbf{\Phi}$ denote the unknown vector

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{\Theta} \\ \mathbf{X}_p \end{pmatrix}$$

where $\mathbf{\Theta}$ stands for the vector of the camera parameters and $\mathbf{X}_p$ denotes the vector of the coordinates of the passpoints. Let $\mathbf{\Theta}^*$ be the vector of the approximate values of the camera parameters and $\Delta\mathbf{\Theta}$ be the vector of corrections to the approximate values of the camera parameters, $\mathbf{X}_p^*$ be the vector of the approximate values of the coordinates of the passpoints, and $\Delta\mathbf{X}_p$ be the vector of corrections to the approximate values of the coordinates of the passpoints. The estimate of the unknown vector $\mathbf{\Phi}$ can be represented as follows:

$$\hat{\mathbf{\Phi}} = \mathbf{\Phi}^* + \Delta\mathbf{\Phi} = \begin{pmatrix} \mathbf{\Theta}^* + \Delta\mathbf{\Theta} \\ \mathbf{X}_p^* + \Delta\mathbf{X}_p \end{pmatrix}$$

where $\mathbf{\Phi}^*$ and $\Delta\mathbf{\Phi}$ denote the vector of the approximate values of the unknown vector $\mathbf{\Phi}$ and the vector of corrections to the approximate values of the unknown, respectively. Let $\mathbf{U}$ be the observation matrix consisting of the coordinates of the 2D image points and $\mathbf{W}$ be the weight matrix of observations. Assuming that the noise is normally distributed, the parameter estimation problem is to determine $\Delta\mathbf{\Phi}$ to minimize

$$\varepsilon = (\mathbf{U} - \mathbf{T}(\mathbf{\Phi}^* + \Delta\mathbf{\Phi}))^T \mathbf{W}(\mathbf{U} - \mathbf{T}(\mathbf{\Phi}^* + \Delta\mathbf{\Phi}))$$

such that $S(\mathbf{\Phi}^* + \Delta\mathbf{\Phi}) = 0$. Here, $S(\mathbf{\Phi}^* + \Delta\mathbf{\Phi}) = 0$ is the constraint equation.

Assuming that the approximate values ($\mathbf{\Theta}^*$ and $\mathbf{X}_p^*$) are available, in order to solve the non-linear least-squares problem, it is sufficient to linearize the objective function and linearize the constraints about the approximate values. When a plane is used to construct a constraint, we should know at least three points lying on the plane. These three points can be control points or passpoints. Their coordinates or approximate coordinates are used for computing the normal vector and signed distance of the plane. Similarly, at least two

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points lying on a line must be known if this line is used for forming a constraint. The above assumptions provide sufficient information for establishing each of the ten constraints independently. Then, $T(\mathbf{\Phi})$ and $S(\mathbf{\Phi})$ are linearized and expressed by a first-order Taylor series expansion, respectively: i.e.,

$$T(\mathbf{\Phi}) = T(\mathbf{\Phi}^*) + \frac{\partial T(\mathbf{\Phi})}{\partial \mathbf{\Phi}_{\Phi^*}} \Delta\mathbf{\Phi} = T(\mathbf{\Phi}^*) + B\Delta\mathbf{\Phi}$$

and

$$S(\mathbf{\Phi}) = S(\mathbf{\Phi}^*) + \frac{\partial S(\mathbf{\Phi})}{\partial \mathbf{\Phi}_{\Phi^*}} \Delta\mathbf{\Phi} = S(\mathbf{\Phi}^*) + H\Delta\mathbf{\Phi}$$

where $\partial T(\mathbf{\Phi})/\partial \mathbf{\Phi}_{\Phi^*}$ and $\partial S(\mathbf{\Phi})/\partial \mathbf{\Phi}_{\Phi^*}$ are the first-order derivatives of $T(\mathbf{\Phi})$ and $S(\mathbf{\Phi})$ with respect to $\mathbf{\Phi}$ at $\mathbf{\Phi}^*$, respectively.

According to the mathematical model and solution of the standard photogrammetric block bundle adjustment presented by Brown *et al.* (1964) and Brown (1968) and considering the constraint equation, the normal equation for the linearized problem is

$$\begin{pmatrix} \mathbf{N}_{\Theta\Theta} & \mathbf{N}_{\Theta X} & \mathbf{H}^{\mathrm{T}} \\ \mathbf{N}_{\Theta X}^{\mathrm{T}} & \mathbf{N}_{XX} & 0 \\ \mathbf{H} & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{\Theta} \\ \Delta\mathbf{X}_p \\ \Delta\mathbf{\Lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{L}_{\Theta} \\ \mathbf{L}_X \\ \mathbf{h} \end{pmatrix}$$

or simply

$$\begin{pmatrix} \mathbf{N} & \mathbf{H}^{\mathrm{T}} \\ \mathbf{H} & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{\Phi} \\ \Delta\mathbf{\Lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{L} \\ \mathbf{h} \end{pmatrix} \tag{1}$$

in which

$$\begin{aligned} \mathbf{N} &= \mathbf{B}^{\mathrm{T}} W \mathbf{B} \\ \mathbf{L} &= \mathbf{B}^{\mathrm{T}} \mathbf{W}(\mathbf{U} - \mathbf{T}(\hat{\Phi})) \\ \mathbf{h} &= -\mathbf{S}(\hat{\Phi}) \end{aligned}$$

where $\Delta\mathbf{\Lambda}$ is the Lagrange multiplier. Finally, the unknown vector $\Delta\mathbf{\Phi}$ is computed by the following equation:

$$\begin{aligned} \Delta\mathbf{\Phi} &= \mathbf{N}^{-1} (\mathbf{I} - \mathbf{H}^{\mathrm{T}} (\mathbf{H}\mathbf{N}^{-1} \mathbf{H}^{\mathrm{T}})^{-1} \mathbf{H}\mathbf{N}^{-1}) \mathbf{L} \\ &\quad + \mathbf{N}^{-1} \mathbf{H}^{\mathrm{T}} (\mathbf{H}\mathbf{N}^{-1} \mathbf{H}^{\mathrm{T}})^{-1} \mathbf{h} \\ &= \mathbf{N}^{-1} \mathbf{P}\mathbf{L} + \mathbf{N}^{-1} \mathbf{H}^{\mathrm{T}} (\mathbf{H}\mathbf{N}^{-1} \mathbf{H}^{\mathrm{T}})^{-1} \mathbf{h} \end{aligned}$$

and the covariance matrix of the unknown is

$$\mathrm{Var}[\Delta\mathbf{\Phi}] = \mathbf{N}^{-1} \mathbf{P}, \tag{2}$$

where $\mathbf{P}$ is the projection matrix.

This paper concentrates on the 3D point estimation in which the camera parameters are regarded as known values and the 3D point coordinates are estimated when using 2D noisy perspective projections. Then the unknown vector $\Delta\mathbf{\Phi}$ is reduced to $\Delta\mathbf{X}_p$ and the normal equation can be simply expressed by

$$\begin{pmatrix} \mathbf{N}_{XX} & \mathbf{H}^{\mathrm{T}} \\ \mathbf{H} & 0 \end{pmatrix} \begin{pmatrix} \Delta\mathbf{X}_p \\ \Delta\mathbf{\Lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{L}_X \\ \mathbf{h} \end{pmatrix} \tag{3}$$

Ten Constraints

We consider ten geometric constraints: (1) coplanarity, (2) collinearity, (3) line to plane angle, (4) line to line angle with intersection, (5) line to line angle without intersection, (6) plane to plane angle with common points, (7) plane to plane angle without common points, (8) distance between points, (9) distance between point and plane, and (10) distance between point and line. The following gives a brief introduction to the mathematical models of the ten constraints. Refer to Huang and Haralick (1997) for more information.

(1) *Coplanarity* is a constraint for the case in which the passpoints must lie on a plane. Suppose that we have n passpoints, $\{(x_i, y_i, z_i);\ i = 1, ..., n\}$, on the plane. The *coplanarity* constraint is expressed by

$$\alpha\Delta x_i + \beta\Delta y_i + \gamma\Delta z_i = h_i \tag{4}$$

where $h_i = -(\alpha x_i + \beta y_i + \gamma z_i + d)$. The plane normal vector $(\alpha, \beta, \gamma)^{\mathrm{T}}$ and the signed distance d are calculated using the approximate coordinates of the n passpoints. The number of constraint equations is equal to the number of points used in the constraint.

(2) *Collinearity* is a constraint for the case in which the passpoints must lie on a line. Suppose that we have n passpoints, $\{(x_i, y_i, z_i);\ i = 1, ..., n\}$, on the line. The *collinearity* constraint equation is written as

$$\begin{pmatrix} 2a_{11} - (a_{21} + a_{31}) \\ 2a_{12} - (a_{22} + a_{32}) \\ 2a_{13} - (a_{23} + a_{33}) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \Delta x_i \\ \Delta y_i \\ \Delta z_i \end{pmatrix} = h_i \tag{5}$$

and h_i is computed by

$$\begin{pmatrix} 2a_{11} - (a_{21} + a_{31}) \\ 2a_{12} - (a_{22} + a_{32}) \\ 2a_{13} - (a_{23} + a_{33}) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} b_x - x_i \\ b_y - y_i \\ b_z - z_i \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= 1 - e_x e_x & a_{12} &= 0 - e_x e_y & a_{13} &= 0 - e_x e_z \\ a_{21} &= 0 - e_y e_x & a_{22} &= 1 - e_y e_y & a_{23} &= 0 - e_y e_z \\ a_{31} &= 0 - e_z e_x & a_{32} &= 0 - e_z e_y & a_{33} &= 1 - e_z e_z \end{aligned}$$

The direction cosines (e_x, e_y, e_z) and the reference point (b_x, b_y, b_z) of the line are calculated using the approximate coordinates of the n passpoints. The number of constraint equations is equal to the number of points used in the constraint.

(3) *Line to plane angle* is for constraining the angle between a line and a plane to be equal to a given angle (ϕ). Let two points (points 1 and 2) lying on the line be unknown points. Then the angle constraint is expressed by

$$\begin{pmatrix} \alpha - \sin(\phi)e_x \\ \beta - \sin(\phi)e_y \\ \gamma - \sin(\phi)e_z \\ \sin(\phi)e_x - \alpha \\ \sin(\phi)e_y - \beta \\ \sin(\phi)e_z - \gamma \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \Delta x_1 \\ \Delta y_1 \\ \Delta z_1 \\ \Delta x_2 \\ \Delta y_2 \\ \Delta z_2 \end{pmatrix} = h, \tag{6}$$

where

$$h = s[\sin(\phi) - (\alpha e_x + \beta e_y + \gamma e_z)].$$

The direction cosines (e_x, e_y, e_z) of the line and the distance s between points 1 and 2 are calculated by the approximate coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) of the two points. The other points are used for calculating the normal vector $(\alpha, \beta, \gamma)^{\mathrm{T}}$ of the plane.

(4) *Line to line angle with intersection* is for constraining the angle between two intersecting lines to be equal to a given angle (ϕ). Let three points be involved in forming the constraint. Referring to Equation 1, the coefficient matrix $\mathbf{H}$ of this constraint equation is written as

$$\begin{pmatrix} s_2\,(e_{x_2} - \cos(\phi)e_{x_1}) \\ s_2\,(e_{y_2} - \cos(\phi)e_{y_1}) \\ s_2\,(e_{z_2} - \cos(\phi)e_{z_1}) \\ s_2\,(e_{x_2} - \cos(\phi)e_{x_1}) + s_1\,(e_{x_1} - \cos(\phi)e_{x_2}) \\ s_2\,(e_{y_2} - \cos(\phi)e_{y_1}) + s_1\,(e_{y_1} - \cos(\phi)e_{y_2}) \\ s_2\,(e_{z_2} - \cos(\phi)e_{z_1}) + s_1\,(e_{z_1} - \cos(\phi)e_{z_2}) \\ s_1\,(e_{x_1} - \cos(\phi)e_{x_2}) \\ s_1\,(e_{y_1} - \cos(\phi)e_{y_2}) \\ s_1\,(e_{z_1} - \cos(\phi)e_{z_2}) \end{pmatrix}^{\mathrm{T}} \tag{7}$$

and the unknown vector $\Delta\mathbf{X}_p$ is

$$(\Delta x_1\ \Delta y_1\ \Delta z_1\ \Delta x_2\ \Delta y_2\ \Delta z_2\ \Delta x_3\ \Delta y_3\ \Delta z_3)^T.$$

Let

$$h = s_1 s_2\,[\cos(\phi) - (e_{x_1}e_{x_2} + e_{y_1}e_{y_2} + e_{z_1}e_{z_2})],$$

where the direction cosines $(e_{x_1}, e_{y_1}, e_{z_1})$ and $(e_{x_2}, e_{y_2}, e_{z_2})$ of the two lines are computed by the coordinates of the passpoints lying on these two lines, respectively. s_1 and s_2 are the distances between points 1 and 2 and between points 2 and 3, respectively. Point 2 is the intersection point.

(5) *Line to line angle without intersection* is for constraining the angle between two non-intersecting lines to be equal to a given angle (ϕ). Let four passpoints be used to form the constraint equation. Each line is determined by two of the four points. The constraint equation is

$$\begin{pmatrix}(1-e_{x_1})e_{x_2}/s_1\\(1-e_{y_1})e_{y_2}/s_1\\(1-e_{z_1})e_{z_2}/s_1\\(e_{x_1}-1)e_{x_2}/s_1\\(e_{y_1}-1)e_{y_2}/s_1\\(e_{z_1}-1)e_{z_2}/s_1\\(1-e_{x_2})e_{x_1}/s_2\\(1-e_{y_2})e_{y_1}/s_2\\(1-e_{z_2})e_{z_1}/s_2\\(e_{x_2}-1)e_{x_1}/s_2\\(e_{y_2}-1)e_{y_1}/s_2\\(e_{z_2}-1)e_{z_1}/s_2\end{pmatrix}^T \begin{pmatrix}\Delta x_1\\\Delta y_1\\\Delta z_1\\\Delta x_2\\\Delta y_2\\\Delta z_2\\\Delta x_3\\\Delta y_3\\\Delta z_3\\\Delta x_4\\\Delta y_4\\\Delta z_4\end{pmatrix} = h, \qquad (8)$$

where

$$h = \cos(\phi) - (e_{x_1}e_{x_2} + e_{y_1}e_{y_2} + e_{z_1}e_{z_2}).$$

(6) *Plane to plane angle with common points* is for constraining the angle between two intersecting planes with a pair of common points to be equal to a given angle (ϕ). Suppose that four passpoints, $\{(x_i, y_i, z_i); i = 1, ..., 4\}$, are used to construct the constraint equation. Points 1 and 2 are located on the intersection line. The coefficient matrix **H** of the constraint is

$$\begin{pmatrix}\delta z_{42}\beta_1 + \delta z_{32}\beta_2 + \delta y_{24}\gamma_1 + \delta y_{23}\gamma_2\\ \delta x_{42}\gamma_1 + \delta x_{32}\gamma_2 + \delta z_{24}\alpha_1 + \delta z_{23}\alpha_2\\ \delta y_{42}\alpha_1 + \delta y_{32}\alpha_2 + \delta x_{24}\beta_1 + \delta x_{23}\beta_2\\ \delta z_{14}\beta_1 + \delta z_{13}\beta_2 + \delta y_{41}\gamma_1 + \delta y_{31}\gamma_2\\ \delta x_{14}\gamma_1 + \delta x_{13}\gamma_2 + \delta z_{41}\alpha_1 + \delta z_{31}\alpha_2\\ \delta y_{14}\alpha_1 + \delta y_{13}\alpha_2 + \delta x_{41}\beta_1 + \delta x_{31}\beta_2\\ \delta z_{21}\beta_2 + \delta y_{12}\gamma_2\\ \delta x_{21}\gamma_2 + \delta z_{12}\alpha_2\\ \delta y_{21}\alpha_2 + \delta x_{12}\beta_2\\ \delta z_{21}\beta_1 + \delta y_{12}\gamma_1\\ \delta x_{21}\gamma_1 + \delta z_{13}\alpha_1\\ \delta y_{21}\alpha_1 + \delta x_{12}\beta_1\end{pmatrix}^T \qquad (9)$$

where

$$\delta x_{ij} = x_i - x_j,\ \delta y_{ij} = y_i - y_j,$$
$$\delta z_{ij} = z_i - z_j,\ i,j = 1,2,3,4.$$

The unknown vector $\mathbf{\Delta X}_p$ is

$$(\Delta x_1\ \Delta y_1\ \Delta z_1, ..., \Delta x_4\ \Delta y_4\ \Delta z_4)^T$$

and

$$h = \cos(\phi) - (\alpha_1^*\alpha_2^* + \beta_1^*\beta_2^* + \gamma_1^*\gamma_2^*),$$

where

$$\begin{pmatrix}\alpha_i^*\\\beta_i^*\\\gamma_i^*\end{pmatrix} = \begin{pmatrix}\alpha_i/n_i\\\beta_i/n_i\\\gamma_i/n_i\end{pmatrix},$$

and

$$n_i = \sqrt{\alpha_i^2 + \beta_i^2 + \gamma_i^2},\ i = 1,2.$$

(7) *Plane to plane angle without common points* is for constraining the angle between two planes to be equal to a given angle (ϕ) in the case that no common points or the common points lying on the intersection line are not provided. Let six points be used to form the constraint equation and each three of them be located on these two planes separately. Let three points (points 1, 2, and 3) lying on the first plane be the unknown points. Then the angle constraint equation is expressed as

$$\begin{pmatrix}\beta_2\delta z_{32} + \gamma_2\delta y_{23}\\ \alpha_2\delta z_{23} + \gamma_2\delta x_{32}\\ \alpha_2\delta y_{32} + \beta_2\delta x_{23}\\ \beta_2\delta z_{13} + \gamma_2\delta y_{31}\\ \alpha_2\delta z_{31} + \gamma_2\delta x_{13}\\ \alpha_2\delta y_{13} + \beta_2\delta x_{31}\\ \beta_2\delta z_{21} + \gamma_2\delta y_{12}\\ \alpha_2\delta z_{12} + \gamma_2\delta x_{21}\\ \alpha_2\delta y_{21} + \beta_2\delta x_{12}\end{pmatrix}^T \begin{pmatrix}\Delta x_1\\\Delta y_1\\\Delta z_1\\\Delta x_2\\\Delta y_2\\\Delta z_2\\\Delta x_3\\\Delta y_3\\\Delta z_3\end{pmatrix} = h, \qquad (10)$$

where

$$\delta x_{ij} = x_i - x_j,\ \delta y_{ij} = y_i - y_j,$$
$$\delta z_{ij} = z_i - z_j,\ i,j = 1,2,3.$$

and

$$h = \cos(\phi) - (\alpha_1^*\alpha_2^* + \beta_1^*\beta_2^* + \gamma_1^*\gamma_2^*).$$

(8) *Distance between points* is for constraining the distance between two passpoints to be equal to a given distance. Suppose that s is the distance between points $p_1(x_1, y_1, z_1)$, $p_2(x_2, y_2, z_2)$. Then the constraint equation is

$$\begin{pmatrix}e_x\\e_y\\e_z\\-e_x\\-e_y\\-e_z\end{pmatrix}^T \begin{pmatrix}\Delta x_1\\\Delta y_1\\\Delta z_1\\\Delta x_2\\\Delta y_2\\\Delta z_2\end{pmatrix} = h, \qquad (11)$$

where

$$e_x = \delta x_{12}/s,\ e_y = \delta y_{12}/s,\ e_z = \delta z_{12}/s$$
$$h = \frac{1}{2}s[1 - (e_x^2 + e_y^2 + e_z^2)].$$

(9) *Distance between point and plane* is for constraining the distance between a point and a plane to be equal to a given distance (s). Assume that point (x_p, y_p, z_p) is the unknown point not lying on the plane $\alpha x + \beta y + \gamma z + d = 0$. The distance constraint is expressed as

$$\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}^T \begin{pmatrix}\Delta x_p\\\Delta y_p\\\Delta z_p\end{pmatrix} = h, \qquad (12)$$

where

$$h = s\sqrt{\alpha^2 + \beta^2 + \gamma^2} - |\alpha x_p + \beta y_p + \gamma z_p + d|.$$

x_p, y_p, z_p are three unknowns in this constraint. The other points are used for calculating the normal vector $(\alpha, \beta, \gamma)^T$ and distance d of the plane.

(10) *Distance between point and line* is for constraining the distance between a point and a line to be equal to a given distance (s). Let point p (not lying on the line) be an unknown point and other points be used for computing the direction cosines of the line. The constraint equation is

$$\begin{pmatrix} x_p - x \\ y_p - y \\ z_p - z \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \Delta x_p \\ \Delta y_p \\ \Delta z_p \end{pmatrix} = h \tag{13}$$

and

$$h = \frac{1}{2}[s^2 - ((x - x_p)^2 + (y - y_p)^2 + (z - z_p)^2],$$

where

$$\frac{x - a}{e_x} = \frac{y - b}{e_y} = \frac{z - c}{e_z} = \lambda$$

and

$$\lambda = \frac{e_x(x_p - a) + e_y(y_p - b) + e_z(z_p - c)}{e_x^2 + e_y^2 + e_z^2},$$

where (a, b, c) are the coordinates of any point located on the line. (e_x, e_y, e_z) are the direction cosines of the line.

Updating Algorithm

The 3D passpoints as unknown points are updated in the camera calibration. For any particular constraint, however, some passpoints are left fixed and then updated. This is to facilitate the iterative computational process. For instance, let the approximate coordinates of five passpoints be given, in which the last four points lie on a plane. In addition, let the distance between the first point and the plane be known. Only the first point is regarded as an unknown point if applying the *constraint distance between point and plane* to these five points. In the iterative process we call the first point a constrained point or constrained estimate. The last four points are termed non-constrained points or non-constrained estimates because their coordinates are not affected by this constraint. The last four passpoints are, however, considered as the unknown points if applying the *coplanarity* constraint to those points. Therefore, a passpoint may play different roles in forming different constraints if it is related to more than two constraints.

Efficiency Evaluation

The following two methods have been used for evaluating the efficiency of the ten constraints: (1) general evaluation and (2) specific evaluation.

General Evaluation

The efficiency of the constraints has been evaluated by analyzing the variances of the 3D passpoints obtained from the camera calibration with and without constraints. The covariance matrix consists of two parts: the unit weight variance (the standard deviation of observations) and the covariance structure matrix. The first one is also called the scalar factor which depends on the accuracy of observations. The second one depends on the geometry of the photogrammetry, consisting of the location of the 3D points, interior orientation elements, and pose of the camera. This method is based on the fact that the structure of the covariance matrices of the 3D passpoints should be improved by adding constraints to the corresponding passpoints in the camera calibration if the mathematical models and the code are correct. In other words, the variances and traces of the covariance structure matrices should become smaller than those before constraining and, correspondingly, the geometry of the photogrammetry should become more reliable. The trace stands for the "total accuracy" of the estimates. Its value is equal to the sum of the diagonal elements or the eigenvalues of the covariance matrix. So this evaluation is not related to noise. The comparison is implemented in four different aspects: (1) single constraint, (2) a group of the same constraints, (3) a combination of different constraints, and (4) an examination of the sum of squared residuals. The sum of squared residuals will, in general, become larger after adding the constraints to the camera calibration because adding a constraint to an estimate will force the estimate generally to be closer to its true value rather than to solve minimizing the unconstrained sum of squared residuals.

The efficiency evaluation of the estimated 3D points is based on the comparison of "relative accuracy" (or relative variance), which is the ratio between the variance derived from the camera calibration with and without constraints. The reason we use the relative accuracy as the estimated quantity for the comparison is because the evaluation is independent of the size of the variances. From Equation 2, the relative variance is expressed as

$$\mathrm{Var}_{\mathrm{Re}}[\Delta \mathbf{X}_p] = \frac{\mathbf{N}^{-1}\mathbf{P}}{\mathbf{N}^{-1}} = \mathbf{P}. \tag{14}$$

Because the coordinates of the 2D image points are measured independently, the projection matrix $\mathbf{P}$ can be partitioned as follows:

$$\mathbf{P} = \mathbf{I} - k^{-1}\,\mathbf{H}^{\mathrm{T}}\,\mathbf{H}\mathbf{N}^{-1}, \tag{15}$$

where

$$\mathbf{I} = \begin{pmatrix} \mathbf{I}_1 & & & \\ & \mathbf{I}_2 & & \\ & & \ddots & \\ & & & \mathbf{I}_n \end{pmatrix},$$

$$\mathbf{H} = (\mathbf{H}_1\ \mathbf{H}_2 \cdots \mathbf{H}_n),$$

$$\mathbf{N} = \begin{pmatrix} \mathbf{N}_1 & & & \\ & \mathbf{N}_2 & & \\ & & \ddots & \\ & & & \mathbf{N}_n \end{pmatrix},$$

$$k = \mathbf{H}_1\mathbf{N}_1^{-1}\,\mathbf{H}_1^{\mathrm{T}} + \cdots + \mathbf{H}_n\mathbf{N}_n^{-1}\mathbf{H}_n^{\mathrm{T}},$$

and in which n is the number of the unknown points in a constraint equation. $\mathbf{I}_i$, $\mathbf{H}_i^{\mathrm{T}}$, and $\mathbf{N}_i$, $i = 1,2,\ldots,n$ are the 3 by 3 unit matrices, 3 by 1 vectors, and 3 by 3 symmetric matrices, respectively. The relative variance matrix of the ith 3D point is expressed by

$$\mathrm{Var}_{\mathrm{Re}}[\Delta \mathbf{X}_{pi}] = \mathbf{I}_i - k^{-1}\mathbf{H}_i^{\mathrm{T}}\mathbf{H}_i\mathbf{N}_i^{-1} \tag{16}$$

and the trace of this variance matrix is equal to

$$3 - k^{-1}\mathrm{Tr}[\mathbf{H}_i^{\mathrm{T}}\mathbf{H}_i\mathbf{N}_i^{-1}], \tag{17}$$

where Tr denotes the trace.

Equation 16 and Expression 17 can be used to evaluate the theoretical accuracy improvement of each unknown point. The larger the value of the second term of Equation 16, the better the accuracy improvement that can be achieved. As far as the effect of a constraint on the accuracy improvement is concerned, the accuracy improvement is mainly determined by k and $\mathbf{H}_i$ because $\mathbf{N}_i$ is not related to the constraints.

$\mathbf{H}_i\mathbf{N}_i^{-1}\mathbf{H}_i^{\mathrm{T}}$, $i = 1,2, \ldots, n$ are positive because the matrices $\mathbf{N}_i^{-1}$, $i = 1,2, \ldots, n$ are positive definite if the vectors $\mathbf{H}_i^{\mathrm{T}}$ are

Table 1. Relative Variance: Single Constraint

No	Pn	Vx/Vx^*	Vy/Vy^*	Vz/Vz^*
1	1	0.88	0.99	3.2×10e-8
	2	0.58	1	3.3×10e-12
	3	0.58	1	9.4×10e-8
	4	0.88	0.99	3.3×10e-11
2	1	0.92	0.54	0.37
	2	0.73	0.65	0.37
	11	0.8	0.6	0.38
3	2	0.58	1	0.82
	6	0.42	1	0.71
4	1	1	0.78	1
	2	1	0.69	1
	4	0.52	1	1
5	1	0.91	1	0.99
	2	0.59	1	0.83
	3	0.59	1	0.55
	4	0.91	1	0.99
6	2	0.28	0.73	0.68
	6	1	1	1
	1	1	1	1
	3	1	1	1
7	2	0.86	1	0.94
	1	0.89	0.99	0.71
	5	0.91	1	0.46
8	1	0.83	1	0.98
	2	0.17	1	0.65
9	1	3.2×10e-12	1	0.88
10	1	0.42	0.8	0.42

Note: The heading No and heading Pn denote constraint number and unknown point number respectively. The heading Vx, Vy, Vz and Vx^*, Vy^*, Vz^* denote the variances of x, y, z of the passpoints obtained from the camera calibration with and without constraints respectively.

non-zero vectors. So k will become smaller when the number of the unknown points in a constraint is reduced, and then the relative variances of the unknown points are decreased. Reviewing Expression 17, the trace of the relative variance matrix of the ith 3D point is reduced when the trace of $\mathbf{H}_i^T \mathbf{H}_i$ becomes smaller. The trace of matrix $\mathbf{H}_i^T \mathbf{H}_i$ is actually the norm of vector $\mathbf{H}_i^T$. Summarizing the above discussion, two *conclusions on* the efficiency evaluation could be made: (1) the efficiency of a constraint is related to the number of unknowns in the constraint equation (the smaller the number, the better the gain in accuracy improvement), and (2) the accuracy of the ith 3D point becomes better with an increase in the norm of the constraint coefficient vector $\mathbf{H}_i^T$. Reviewing constraints from 1 to 10, the *coplanarity* and *collinearity* constraints contain only one unknown point, the distance-type constraints (constraint numbers 8 to 10) also have only one unknown point except that constraint 8 has two unknown points, and the number of the unknown points in the angle-type constraints (constraint numbers 3 to 7) is equal to or larger than three except that constraint 3 has two unknown points. The norms of $\mathbf{H}_i^T$ are used to assess the efficiency of constraints with the same number of unknown points. The efficiencies of the *coplanarity* and *distance between point and plane* constraints are the same if these two constraints are related to the same plane. The *collinearity* constraint is not as strong as the former two constraints. In general, the *coplanarity, distance between point and plane*, and *collinearity* constraints are the strongest, the distance-type constraints except constraint 8 are second, and the angle-type constraints are the weakest.

The above discussion gives us a general idea about the efficiency of the ten constraints. The following numerical example has been used to assess the efficiency of the ten constraints in a particular case. The findings in this example numerically verify the correctness of the two theoretical conclusions we have made. In addition, some new ideas obtained from this example can be used to improve the accuracy of the 3D points when applying the ten constraints to real data. Two images with 20 3D points have been employed in the example. The interior and the exterior orientation elements of the two images, coordinates of the image points, and coordinates of the 3D points are provided in the Appendix. It should be pointed out that, regardless of the units in Tables A1 to A3, the results of the efficiency evaluation are not affected.

Table 1 shows the accuracy comparison based on the relative variance. A relative variance that equals 1 indicates no accuracy improvement. The following is the explanation of Table 1 (refer to Figure 2 for facilitating the explanation):

- No. 1: The *coplanarity* constraint is used to force passpoints 1, 2, 3, and 4 to lie on the plane 1-2-3-4, consisting of these four passpoints. The reason why the relative variances in z are much smaller than the others is because the normal vector of plane 1-2-3-4 is parallel to the z axis. The next section will give a more detailed explanation of this.
- No. 2: The *collinearity* constraint is used to force passpoints 1, 2, and 11 to lie on the line consisting of these three passpoints.
- No. 3: The *line to plane angle* constraint is used to constrain the angle between line 2-6, determined by passpoints 2 and 6, and plane 1-2-3-4, determined by points 1, 2, 3, and 4, to equal to 90°. Points 2 and 6 are two unknown points in the constraint.
- No 4: The *line to line angle with intersection* constraint is used to constrain the angle between line 1-2 and line 1-4 to equal 90°. Point 1 is the intersection point. These three points are the unknown points.
- No. 5: The *line to line angle without intersection* constraint is used to constrain line 1-2 to be parallel to line 4-3. These four points are the unknown points.
- No. 6: The *plane to plane angle with common points* constraint is employed to constrain the angle between planes 1-2-6 and 3-2-4 to equal 90°. Points 2 and 6 are two common points lying on the intersection line between these two planes. The reason why the relative variances of point 2 are much smaller than those of the other points is because point 2 is the intersection point of three lines 1-2, 3-2, and 6-2. From a geometrical point of view, the influence of the constraint on improving the accuracy of point 2 is much larger than the others.
- No. 7: The *plane to plane angle without common points* constraint is employed to constrain plane 2-1-5 to be parallel to plane 3-4-8. Points 1, 2, and 5 are three unknown points. The other three points are the non-constrained points for determining plane 3-4-8.
- No. 8: The *distance between points* constraint is used to force the distance between points 1 and 2 to equal a given distance (14.7043). These two points are the unknown points.
- No. 9: The *distance between point and plane* constraint is used to force the distance between point 1 and plane 2-3-7-6 to equal a given distance (14.7043). Only point 1 is the unknown point. Apparently, the x axis is parallel to the normal vector of the plane. So the relative variance σ_x/σ_{x^*} is much smaller than the others.
- No. 10: The *distance between point and line* constraint is used to force the distance between point 1 and line 2-12-3 to be equal to a given distance (14.7043). Point 1 is the unknown point. This constraint and the *distance between point and plane* constraint are the distance-type constraints with the same major direction except that the geometrical "basis" (plane and line) are different. The reason why the relative

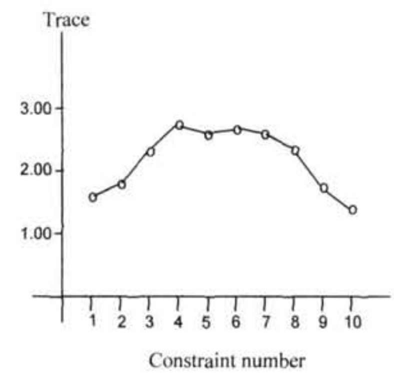

Figure 1. The trace of constrained points.

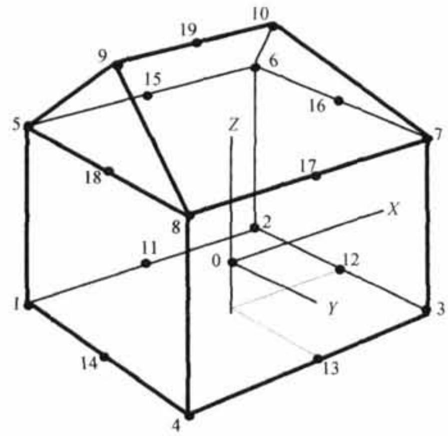

Figure 2. The structure of 3D points.

variance of point 1 in the major direction derived from the *distance between point and plane* constraint is much smaller than that obtained from the *distance between point and line* constraint is because the basis in the first constraint is much stronger than the one in the second constraint from a geometrical point of view.

The traces illustrated in Figure 1 are the so-called average standardized traces. They are the average values of the sum of the relative variances of the constrained points. Trace equals 3 indicates no accuracy improvement in the estimated points. Reviewing Table 1 and Figure 1, we can make the same conclusions as we have made in the theoretical research above.

Our experiment shows that the efficiency of the constraints has been promoted by grouping the same types of constraints and by combining different types of constraints. Although the sum of squared residuals is increased with the number of constraints in the camera calibration, the variance of the estimated parameter is decreased. From Figure 1, it is apparent that a point which is involved in a few constraints can only have a reduction in variance by an order of magnitude. Refer to Huang and Haralick (1997) for more information.

Specific Evaluation

Specific evaluation means that assessing accuracy improvement is conducted in "the major direction of accuracy improvement." Figure 2 shows that the axes of the 3D coordinate system are parallel to the edges of the house, which consists of 20 3D points. The advantage of such an arrangement is to provide a method to predict the largest reduction in the variances $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$, and $\boldsymbol{\sigma}_z$ by adding a constraint to the camera calibration. For instance, let us assume that the distance between points p_1 and p_2 is known. The constraint distance between points is adopted for constraining the distance between these two points to be equal to the known distance. It is easy to see from Figure 3 that the distance is actually the difference of the x coordinates between the two points. It could be expected that the $\boldsymbol{\sigma}_x$ should be reduced much more than the $\boldsymbol{\sigma}_y$ and $\boldsymbol{\sigma}_z$ when applying the constraint to the two points. Therefore, the x axis is called the major direction of accuracy improvement. Table 1 shows apparently that the accuracy improvement in the major direction is much larger than in the other two directions.

Correctness Test

The statistical test for the correctness of the camera calibration with constraints is based on the fact that the distribution of a linear combination of the normally distributed variables is also normal. This test is not related to the constraints. It is associated with the type of noise to be added in the camera calibration for generating the simulated sample data. The basic assumptions for the test are (1) that the mathematical model employed can be linearized and that the scalar function (objective function) has finite second partial derivatives and that the random perturbations are small enough so that the relationship between the scalar function evaluated at the ideal but unknown input and output quantities and the observed input quantity and perturbed output quantity can be approximated sufficiently well by a first-order Taylor series expansion; and (2) that the noise we add in the 2D image points is an additive random perturbation and that the estimated quantity produced by the camera calibration processing is also an additive random perturbation (Haralick, 1993; Anderson, 1984). The coordinates of the 3D passpoints and their covariances computed by the camera calibration without introducing noise to the 2D image are regarded as the population mean and population covariance of the 3D points. Let $\mu = \Theta$ be the population mean and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_\Theta$ the population covariance. The sample mean and sample covariance can be obtained by adding random noise to the 2D data in the camera calibration process. Let m be the number of replications. From each replication, the result is a sample mean $\bar{x} = M_r$ and a sample covariance $\mathbf{S} = \mathbf{S}_r$ where

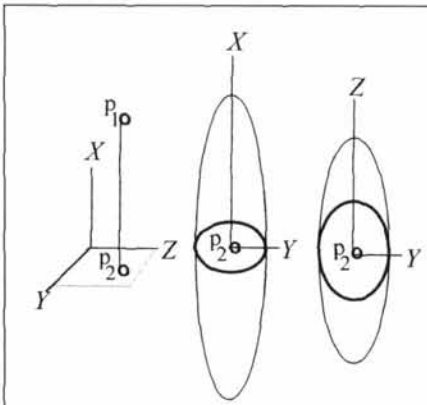

Figure 3. The major direction of accuracy improvement.

TABLE 2. FIVE STATISTICAL TESTS

1. Test 1:	$\mu = \mu_o$	with known	$\Sigma = \Sigma_o$
2. Test 2:	$\mu = \mu_o$	with known	Σ
3. Test 3:	$\Sigma = \Sigma_o$	with known	μ
4. Test 4:	$\Sigma = \Sigma_o$	with known	$\mu = \mu_o$
5. Test 5:	$\Sigma = \Sigma_o$	and	$\mu = \mu_o$

$$M_r = \frac{1}{n}\sum_{i=1}^{n} \Theta_{r_i},$$
$$\mathbf{S}_r = \frac{1}{n}\sum_{i=1}^{n} (\Theta_{r_i} - M_r)(\Theta_{r_i} - M_r)^{\mathrm{T}},$$

in which n is the number of samples. Let us assume that the linearization was valid in the neighborhood of the estimate. The distribution of the sample estimates should be normally distributed if normal noise is added to the 2D points in the data processing. Then it is appropriate to test whether that is the case. Five tests (Table 2) can be conducted if a sample of data comes from a multivariate $N(\mu, \sigma)$ distribution (Kanungo and Haralick, 1995). The tests differ in what is assumed to be unknown.

One-hundred replications, and 100 trials for each replication, have been generated. Because the population mean and population covariance matrix of the 3D points are known, we choose "test 5" for testing the models used in the constraints. In this case, the question is whether or not the sample is from a Gaussian population whose mean is μ_o and covariance is Σ_o. The statistical test consists of two steps: (1) check how many samples pass the test, and (2) examine the probability of each passpoint. Normally, about 95 percent (not less than 90 percent) of the samples should pass the test if significance level $\alpha = 0.05$ is selected. The probability is important information to determine whether the estimate passes the test or not. If the probability is larger than 0.05, we pass the test.

Statistical Test of the Camera Calibration without Constraints

Before testing the camera calibration with constraints, the camera calibration without constraints should pass the test. Table 3 shows the test result of the camera calibration without constraints. One-hundred replications, and 100 trials for each replication, have been used in the test. The heading n% denotes the percentage of samples passing the test and heading Prob denotes the probability. The numerical results show that every point passes the test because the percentages of samples passing the test are larger than 90 percent and the probabilities are larger than 0.05.

Statistical Test of the Camera Calibration with Constraints

The constraints play a role in forcing the constrained estimate to converge to its true value so that its variance becomes smaller. Particularly, its covariance converges to zero if the major direction is parallel to the axes of the coordinate system. On the other hand, the constraints lead to a linear correlation between the constrained estimates. As a result, the determinants of their covariance matrices are equal to zero. This is the reason that the constrained estimates fail to pass the test.

Test in the Case that the Major Direction Is Parallel to One of the Coordinate Axes

Table 4 shows the test result of the camera calibration with two *coplanarity* constraints. One-hundred replications, and 100 trials for each replication, have been used in this test. Points 1, 2, 3, 4 and points 5, 6, 7, 8 lying on two separate horizontal planes (Figure 2) are used in these two constraints. The eight points are not successful in passing the test because the determinants of their covariance matrices are equal to zero. In order to solve the problem in the test, consider the difference between the population mean and sample mean of one of the eight constrained points (d_x, d_y, and d_z of point 1): i.e.,

$$1.27 \times 10^{-03}\ 9.1 \times 10^{-05}\ 4.1 \times 10^{-17}.$$

This shows that the difference in z coordinates is close to zero, which means that the *coplanarity* constraints make the sample mean converge to the population mean in the major direction of accuracy improvement because the major direction is parallel to the z axis (Figure 2). This leads to the corresponding variance converging to zero in the major direction. The covariance matrices of point 1 in three different cases are shown as follows.

The population covariance matrix of point 1 in the non-constrained case.

$$\begin{matrix} 1.38 \times 10^{-06} & 5.79 \times 10^{-08} & 6.48 \times 10^{-07} \\ 5.79 \times 10^{-08} & 1.74 \times 10^{-06} & 1.78 \times 10^{-07} \\ 6.48 \times 10^{-07} & 1.78 \times 10^{-07} & 2.54 \times 10^{-06} \end{matrix}$$

The population covariance matrix of point 1 constrained by a *coplanarity* constraint.

$$\begin{matrix} 1.21 \times 10^{-06} & 1.25 \times 10^{-08} & 5.98 \times 10^{-15} \\ 1.25 \times 10^{-08} & 1.73 \times 10^{-06} & 1.64 \times 10^{-15} \\ 5.98 \times 10^{-15} & 1.64 \times 10^{-15} & 2.34 \times 10^{-14} \end{matrix}$$

TABLE 3. STATISTICAL TEST: CAMERA CALIBRATION WITHOUT CONSTRAINTS

Pn	n%	Prob	Pn	n%	Prob
1	91	0.28	11	92	0.07
2	93	0.36	12	92	0.06
3	96	0.85	13	93	0.6
4	96	0.86	14	91	0.67
5	97	0.48	15	98	0.43
6	95	0.07	16	99	0.34
7	95	0.82	17	94	0.85
8	94	0.17	18	93	0.53
9	95	0.67	19	95	0.75
10	97	0.31	20	94	0.94

TABLE 4. STATISTICAL TEST: CAMERA CALIBRATION WITH TWO COPLANARITY CONSTRAINTS

Pn	n%	Prob	Pn	n%	Prob
1	0	0	11	92	0.07
2	0	0	12	92	0.06
3	0	0	13	93	0.62
4	0	0	14	91	0.67
5	0	0	15	98	0.43
6	0	0	16	99	0.35
7	0	0	17	94	0.85
8	0	0	18	93	0.54
9	95	0.63	19	95	0.75
10	97	0.32	20	94	0.95

The sample covariance matrix of point 1 constrained by a *coplanarity* constraint.

$$\begin{matrix} 1.26 \times 10^{-06} & 1.16 \times 10^{-07} & 1.10 \times 10^{-32} \\ 1.16 \times 10^{-07} & 1.38 \times 10^{-06} & -2.89 \times 10^{-32} \\ 1.10 \times 10^{-32} & -2.89 \times 10^{-32} & 7.97 \times 10^{-31} \end{matrix}$$

The determinants of the three covariance matrices above are

$$5.31 \times 10^{-18} \quad 4.90 \times 10^{-26} \quad 1.38 \times 10^{-42}$$

Analyzing the above three covariance matrices and their determinants, two points can be summarized as follows: (1) an obvious accuracy improvement in the major direction has been achieved by applying the *coplanarity* constraint to the eight constrained points; and (2) the convergence speed to zero of the sample variance in the major direction is much faster than that of the population variance. Correspondingly, the determinant of the sample covariance matrix converges to zero much faster than that of the population covariance matrix. The zero determinant indicates that the covariance matrix is singular. In this particular case, the major direction is also the direction in which the estimates are linearly correlated because the zero sample variance only happens to the constrained estimate. This provides an efficient way to solve the singular problem encountered in testing camera calibration with constraints. That is to say, removing the estimates with zero sample variance is the way of removing the correlated estimates from the test data. Applying this new method to point 1, the population and the sample covariance matrices of point 1 become 2 by 2 matrices shown as follows:

$$\begin{matrix} 1.21 \times 10^{-06} & 1.25 \times 10^{-08} \\ 1.25 \times 10^{-08} & 1.73 \times 10^{-06} \end{matrix}$$

and

$$\begin{matrix} 1.26 \times 10^{-06} & 1.16 \times 10^{-07} \\ 1.16 \times 10^{-07} & 1.38 \times 10^{-06} \end{matrix}$$

and their determinants are

$$2.10 \times 10^{-12} \quad 1.74 \times 10^{-12}$$

These two determinants are not close to zero, which means that the singular problem has been solved. Table 5 lists the probabilities of the eight constrained passpoints by using the new method. The numerical results indicate that the eight points pass the test.

Test in the Case that the Major Direction Is not Parallel to the Coordinate Axes

It is true that the determinate of the covariance matrix of the constrained point is equal to zero due to the correlation between its components (x, y, and z) derived from constraints. The variances of the estimated components may not necessarily be equal to zero if the major direction is not parallel to the coordinate axes. In order to solve the zero determinant problem using the proposed method in this case, applying an orthogonal transformation to the population mean and covariance, the sample mean and covariance is required: i.e.,

$$\begin{aligned} \boldsymbol{\mu}_y &= \boldsymbol{\mu}_x \mathbf{T} \\ \boldsymbol{\sigma}_y &= \mathbf{T}^T \boldsymbol{\Sigma}_x \mathbf{T} \\ \bar{\mathbf{y}} &= \bar{\mathbf{x}} \mathbf{T} \\ \mathbf{S}_y &= \mathbf{T}^T \mathbf{S}_x \mathbf{T}, \end{aligned}$$

where matrix $\mathbf{T}$ is the orthogonal transformation matrix. The advantage of the orthogonal transformation is that the property of the matrix may not be changed by the transformation. In other words, the trace of the covariance matrix may not be affected by the transformation due to the symmetry of the covariance matrix. So the results of efficiency evaluation based on the transformed covariance matrix are the same as those obtained from the original covariance matrix. Three methods are presented as follows. Let $\mathbf{X}$ be the coordinate matrix of the constrained points in the original coordinate system and $\mathbf{Y}$ be a new coordinate matrix in which the major direction is parallel to one of the axes in the old coordinate system. The first method for calculating the transformation matrix $\mathbf{T}$ is written as

$$\mathbf{T} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{Y}.$$

The second method is more complicated than the first one. The general formula for the 3D coordinate transformation is expressed by

$$\mathbf{Y}_i = \Delta\mathbf{X} + k\mathbf{T}\mathbf{X}_i, \ (i = 1,\ldots,n)$$

where $\Delta\mathbf{X}$ is the translation vector, n is the number of the constrained points, and $k = 1$ (the scale factor). The objective function for computing $\mathbf{T}$ is

$$\sum_{i=1}^{n} [(\mathbf{T}\mathbf{X}_i + \Delta\mathbf{X}) - \mathbf{Y}_i]^T = \min,$$

in which $\Delta\mathbf{X}$ is equal to zero if $\mathbf{X}$ and $\mathbf{Y}$ are centralized. Then the objective function can be simplified as

$$\sum_{i=1}^{n} (\mathbf{T}\mathbf{X}_i^{\circ} - \mathbf{Y}_i^{\circ})^2 = \min$$

where $\mathbf{X}^{\circ}$ and $\mathbf{Y}^{\circ}$ are the centralized coordinates. The transformation matrix $\mathbf{T}$ is computed as follows:

$$\mathbf{D}_{3 \times 3} = \mathbf{X}^{\circ}_{3 \times n}{}^{T}\mathbf{Y}^{\circ}_{n \times 3}$$

and matrix $\mathbf{D}$ is decomposed into (the singular value decomposition)

$$\mathbf{D}_{3 \times 3} = \mathbf{U}_{3 \times 3}\mathbf{S}_{3 \times 3}\mathbf{V}^T_{3 \times 3}.$$

The transformation matrix is the product of matrices $\mathbf{U}$ and $\mathbf{V}^T$: i.e.,

$$\mathbf{T} = \mathbf{U}\mathbf{V}^T.$$

The third method is different from the above two methods. The transformation matrix consists of the eigenvectors derived from the covariance matrix of the constrained point. After transformation, the covariance matrix becomes a diagonal matrix in which one of its diagonal elements is equal to zero.

We suggest using the first two methods if one knows the relationship between the major direction and the coordinate axes. The population mean and sample mean remain unchanged in the first method and remain partly unchanged in the second method after transformation because these two methods are not involved in translation. The statistical test will be simplified if we can keep the population mean and sample mean unchanged in the transformation. The third method can be applied in the case that we have no idea about the relationship between the major direction and the

Table 5. The Probabilities of Eight Constrained Points Using the New Test Method

Pn	Prob	Pn	Prob	Pn	Prob
1	0.08	2	0.06	3	0.39
4	0.21	5	0.92	6	0.18
7	0.72	8	0.1		

axes, but the values of the means will be changed after the transformation.

Independence Condition

The ten constraints can be used separately and simultaneously. To achieve a desired efficiency, a good combination of the constraints in terms of the geometric structure of the 3D points is suggested. The constraints having high efficiency should have a high priority to be considered to be used in the camera calibration process. Before applying the constraints to the camera calibration, the constraints should satisfy the independence condition. For instance, distances from a passpoint to several other points are given. The maximum number of the *distance between points* constraints that can be applied to this passpoint is three. Adding more than three constraints to the passpoint will lead to an over-constrained problem. As a result, the rank of the coefficient matrix of the constraints will not be full.

Conclusion

This paper discusses using ten constraints for improving the accuracy of the 3D passpoints. The efficiency and correctness of the constraints are two major problems in the discussion. The constraints play a role in forcing the constrained estimate to converge to its true values in the camera calibration. As a result, the variance of the constrained estimate converges to zero in the major direction. The zero variance strongly proves that the corresponding constraints are efficient and also verifies the correctness of the models being used in the construction of the constraints. The zero determinant problem will be encountered in testing the correctness of models when adding the constraints to the camera calibration. This problem can be solved by removing the estimate with zero variance from the test data. Removing the constrained estimates from the data does not influence correctness of the test. The methods presented in this paper can be used to test the correctness of any software as well as the efficiency of the mathematical models used in the software.

The mathematical models used in the paper can be easily modified (for instance, by adding or deleting some of the unknowns in the constraint equations) to be suitable for a real application. In addition to the accuracy improvement of the estimated 3D points, the constraints can also be used to solve the rank-defect problem resulting when the number of the control points is not enough for determining the unknowns.

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Appendix

TABLE A1. INTERIOR AND EXTERIOR ORIENTATIONS OF TWO IMAGES

Item	Image 1	Image 2
f	1500	1500
x	0	0
y	0	0
X	−1.5	−1.5
Y	−1.5	−1.5
Z	−2.1213203	−2.1213203
a	0.1464466	−0.1464466
b	−0.3535534	−0.3535534
c	−0.3535534	0.3535534
d	0.8535534	0.8535534

Note: f, x, and y are the interior orientation elements (the focal length and the coordinates of the principle points); X, Y, and Z are the exterior coordinates of the perspective center; and a, b, c, and d are the elements constituting the rotation matrices (**R**): i,e.,

$$\mathbf{R} = \begin{pmatrix} d^2 + a^2 - b^2 - c^2 & 2\,(ab - cd) & 2ac + bd) \\ 2(ab + bd) & d^2 - a^2 + b^2 - c^2 & 2(bc - ad) \\ 2(ac - bd) & 2(bc - ad) & d^2 - a^2 - b^2 + c^2 \end{pmatrix}$$

The coordinates of the 2D image points listed in Tables A2 and A3 are needed to transform to the camera frame system. The transformation equation is

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 500 \\ -500 \end{pmatrix},$$

where x_c and y_c are the coordinates of the 2D points used in the camera calibration process.

TABLE A2. COORDINATES OF 20 3D POINTS

Pn	X	Y	Z
1	−0.735215	−0.441129	−0.51465
2	0.735215	−0.441129	−0.51465
3	0.735215	0.441129	−0.51465
4	−0.735215	0.441129	−0.51465
5	−0.735215	−0.441129	0.220564
6	0.735215	−0.441129	0.220564
7	0.735215	0.441129	0.220564
8	−0.735215	0.441129	0.220564
9	−0.294086	0	0.51465
10	0.147043	0	0.51465
11	0	−0.441129	−0.51465
12	0.735215	0	−0.51465
13	0	0.441129	−0.51465
14	−0.735215	0	−0.51465
15	0	−0.441129	0.220564
16	0.735215	0	0.220564
17	0	0.441129	0.220564
18	−0.735215	0	0.220564
19	−0.735215	0	0.51465
20	0	0	0

TABLE A3. COORDINATES OF 20 2D POINTS

Image 1			Image 2		
Pn	x	y	Pn	x	y
1	6.5231356	6.6425934	1	−0.01277	3.6930557
2	0.5169144	2.2461469	2	5.9674324	0.5706829
3	4.0325676	0.5706829	3	9.4830856	2.2461469
4	10.012772	3.6930557	4	3.4768643	6.6425934
5	6.2147616	9.3469338	5	0.8533244	6.5105355
6	1.2225273	5.0405054	6	5.8331039	3.2684527
7	4.1668961	3.2684527	7	8.7774727	5.0405054
8	9.1466756	6.5105355	8	3.7852384	9.3469338
9	5.9696548	7.3825438	9	4.0303452	7.3825438
10	4.5462818	6.2671851	10	5.4537182	6.2671851
11	3.0629952	4.1098325	11	3.3621107	1.9309685
12	2.4038201	1.3468994	12	7.5961799	1.3468994
13	6.6378893	1.9309685	13	6.9370048	4.1098325
14	8.4376026	5.0244326	14	1.5623974	5.0244326
15	3.4060492	6.9240669	15	3.6142922	4.7130089
16	2.7868658	4.0990166	16	7.2131342	4.0990166
17	6.3857078	4.7130089	17	6.5939508	6.924067
18	7.7966788	7.8165523	18	2.2033212	7.8165523
19	5.2343786	6.8063795	19	4.7656214	6.8063795
20	5	5	20	5	5

Note: The coordinates are the numbers listed in Table A3 multiplied by 100.